Trigonometry Through Wayfinding and Navigation Across the Pacific

Trigonometry Through Wayfinding and Navigation Across the Pacific

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For Tasha and Kamuela

Acknowledgements

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Preface

About this book: This book integrates navigation into trigonometry curriculum.

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Chapter 1 is available to down as a PDF at https://www.kamuelayong.com/Trigonometry-Yong-Chapter-1.pdf

Core Standards

The curriculum in this book aligns with the Common Core State Standards: corestandards.org[2]

High School: Functions: Trigonometric Functions corestandards.org[3]
Extend the domain of trigonometric functions using the unit circle.
High School: Functions: Trigonometric Functions
Extend the domain of trigonometric functions using the unit circle

- CCSS.MATH.CONTENT.HSF.TF.A.1[4]: Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. (Definition 1.2.18)

- CCSS.MATH.CONTENT.HSF.TF.A.2[5]: Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (Subsection 1.3.2)

- CCSS.MATH.CONTENT.HSF.TF.A.3[6]: Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for x, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number. (Subsection 1.4.2, Subsection 1.5.3)

- CCSS.MATH.CONTENT.HSF.TF.A.4[7]: Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions. Model periodic phenomena with trigonometric functions. (Subsection 1.5.4, Subsection 1.5.7)

Prove and apply trigonometric identities

- CCSS.MATH.CONTENT.HSF.TF.C.8[8]: Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ and use it to find $\sin\theta$, $\cos\theta$, or $\tan\theta$ given $\sin\theta$, $\cos\theta$, or $\tan\theta$ and the quadrant of the angle. (Definition 1.5.20)

High School: Geometry: Similarity, Right Triangles, and Trigonometry
Define trigonometric ratios and solve problems involving right triangles

[2] http://www.corestandards.org/Math/Content/HSF/TF/
[3] http://www.corestandards.org/Math/Content/HSF/TF/
[4] www.thecorestandards.org/Math/Content/HSF/TF/A/1/
[5] www.thecorestandards.org/Math/Content/HSF/TF/A/2/
[6] www.thecorestandards.org/Math/Content/HSF/TF/A/3/
[7] www.thecorestandards.org/Math/Content/HSF/TF/A/4/
[8] www.thecorestandards.org/Math/Content/HSF/TF/C/8/

- CCSS.MATH.CONTENT.HSG.SRT.C.6[9]: Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (Definition 1.4.1)
- CCSS.MATH.CONTENT.HSG.SRT.C.7[10]: Explain and use the relationship between the sine and cosine of complementary angles. (Definition 1.4.7, Remark 1.4.9)
- CCSS.MATH.CONTENT.HSG.SRT.C.8[11]: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. (Subsection 1.4.6)

High School: Geometry : Circles
Find arc lengths and areas of sectors of circles

- CCSS.MATH.CONTENT.HSG.C.B.5[12]: Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. (Definition 1.2.18, Theorem 1.2.25, Definition 1.2.28)

[9] www.thecorestandards.org/Math/Content/HSG/SRT/C/6/
[10] www.thecorestandards.org/Math/Content/HSG/SRT/C/7/
[11] www.thecorestandards.org/Math/Content/HSG/SRT/C/8/
[12] www.thecorestandards.org/Math/Content/HSG/C/B/5/

Contents

Chapter 1

Trigonometry

1.1 Pacific Island Navigation

Pacific Islanders have been navigating across long distances in the Pacific for centuries, even before the use of magnetic compasses or modern instruments. They relied on observations of celestial bodies, such as stars and the sun, as well as natural elements like ocean patterns, winds, bird behaviors, and other environmental cues to determine their position relative to known landmarks such as islands, reefs, and continents. Over time, much of this traditional navigational knowledge was lost in many parts of the Pacific. However, some islands, particularly in Micronesia and on Taumako Island in the Solomon Islands, managed to preserve the art and science of traditional navigation. These places continued to uphold the practice, teaching new generations to build ocean-going canoes and develop navigational skills based on profound knowledge of the natural world.

1.1.1 Micronesia

Navigators in Micronesia utilize the *paafu* mat or map (shown in Figure 1.1.1). It is often misunderstood and misinterpreted as a Star "Compass" due to its use of stars and constellations for direction finding. However, paafu serves a different purpose and is not equivalent to the cardinal directionality marked by compasses (North, South, East, and West). Instead, it is a learning and teaching tool designed to teach the locational positions of islands, locales, or canoes relative to other places. This is achieved by observing the rising and setting points of stars and constellations, which act as markers for different locations.

Figure 1.1.1 Paafu mat or map - Photo courtesy of Kānehūnāmoku Voyaging Academy

A constellation is a cluster of stars whose shapes and meanings reflect and carry cultural significance. In modern society, a well-known cluster of stars in the southern hemisphere, shaped like a cruciform, is commonly referred to as the "Southern Cross." However, for the Polowatese and other islanders from the Central Carolines region in the western Pacific islands, this same constellation resembles the triggerfish, and so it is named accordingly.

In the Central Carolines, the general location in the celestial sky where stars appear to rise after sundown is referred to as "*tan*." This term is often mistakenly translated as "east" due to the modern association of stars (like the sun) with "rising" in the east. However, it's important to note that "tan" means "rising" and not "eastward."

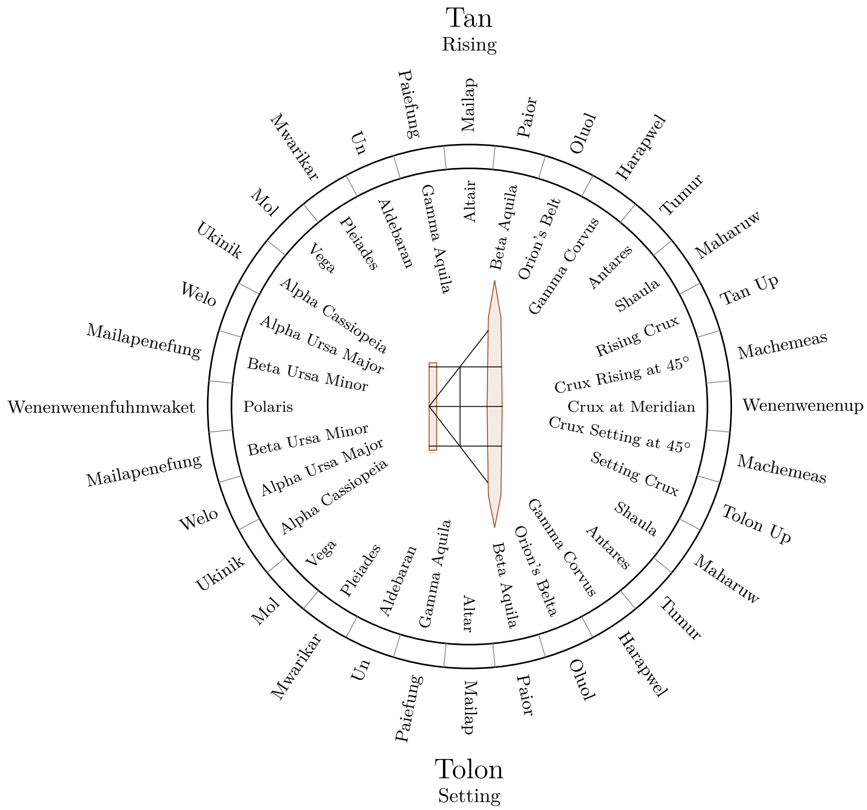


Figure 1.1.2 The paafu, or Micronesian Star Compass. Stars are identified using the Polowat dialect of the Chuukese language as it is used by members of the Weriyeng School of navigation.

Figure 1.1.2 orients the cardinal direction known as "east" at the top of the page, and so the top half of this diagram is also identified as "tan" – where stars appear to rise. The diagram illustrates the apparent path of stars across the sky each night and day (though most stars are not visible during the day) and throughout a year. In this system, the rising and presence of specific stars mark months, and these same stars will eventually set at a point horizontally opposite to where they rose. This point corresponds to the cardinal direction known as "west," which is referred to as "*tolon*" – the area where the stars "set" or go down.

In the paafu "map" shown in Figure 1.1.2, a canoe is placed at the center, and the star called Mailap (Altair) marks due east. The time and location when Mailap rises are referred to as "Tan Mailap" (rising Mailap), while the time and place it sets are referred to as "Tolon Mailap" (setting Mailap). The map's orientation places east, or the rising points of the navigation or paafu stars, at the top of the circle, and west, or the setting points of these stars, at the bottom of the circle. As a map, paafu uses the rising and setting points of stars to mark places around a given locale, which is placed at the center of the circle. Table 1.1.3 displays the star and constellation names, provided in both Polowat and according to the International Astronomical Union, listed in the order of their rising during the third week of March in Polowat.

Table 1.1.3 Star and constellation names in Polowat

Polowat	International Astronomical Union
Wenenwenenfuhmwaket	Polaris (always above the horizon)
Tan Mwarikar [Mahrah-ker]	Pleiades aka Seven Sisters
Tan Un [Oon]	Aldebaran
Tan Uliul [Ooh-lee-ool]	Orion's Belt
Tan Harapwel [Ah-rah-pwol]	Gamma Corvus
Tan Mailapenefang [My Lap in a Fang]	Beta Ursa Minor in Big Dipper
Tan Up [Oop]	Crux or Southern Cross at Rising
Machemeas [Matche-may-ess]	Crux or S. Cross at 45° [1]
Tan Welo [Well-Ah]	Alpha Ursa Major in Big Dipper
Wenenwenenup [Wehneh wehnen Oop]	Crux or S. Cross at Meridian or upright
Tan Tumur [Two More]	Antares or Scorpio's tail
Tan Maharuw [Maa-Haa-Roo]	Shaula or Scorpio's stinger
Tan Mol [Mohl]	Vega
Tan Mailap [My Lap]	Altair
Tan Paiefung [Pie Efung]	Gamma Aquila
Tan Paior [Pie Or]	Beta Aquila
Tan Ukinik [Icky Nick]	Cassioepea

Paafu can also be used to identify the direction in which moving objects, such as canoes, or creatures like birds, fish, and humans, are heading or coming from. This version of paafu utilizes the Polowat dialect of the Chuukese language, as used by members of the Weriyeng School of navigation.

1.1.2 Hawai‘i

With the aim of reviving wayfinding in Hawai‘i, Nainoa Thompson journeyed to the island of Satawal in the Federated States of Micronesia to learn from master navigator Mau Piailug, affectionately known as Papa Mau. Using this knowledge, Thompson adopted the paafu method, leading to the creation of the Hawaiian Star Compass, also referred to as the *Kūkuluokalani* (Figure 1.1.4).

In the star compass, featuring the figure of an ‘iwa or great frigatebird at its center, Thompson divides the visual horizon into 32 equidistant points around a circle, referred to as houses. Each house in the Hawaiian Star Compass represents a specific space on the horizon (11.25°) where celestial bodies such as the sun, stars, moon, and planets rise and set. In the same way that we use addresses to locate homes, each celestial body has its own address represented by these houses.

[1] The "Tan" prefix is not used for this position, because it is no longer rising

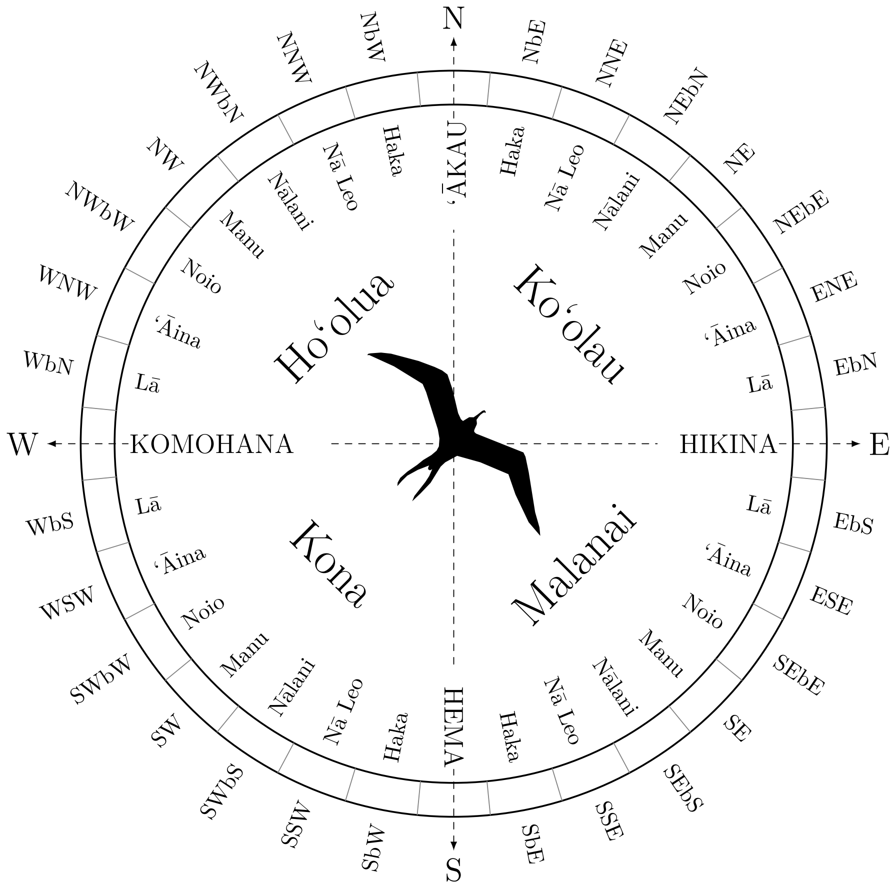


Figure 1.1.4 Hawaiian Star Compass, also known as the Kūkuluokalani.

The four cardinal points align with particular houses. Stars rise from the horizon called *Hikina* ("To Arrive") or East and set on the horizon called *Komohana* ("To Enter") or West. If you face Komohana (West) with your back towards Hikina (East), your right will point towards *ʻĀkau* ("Right") or North, and your left will point towards *Hema* ("Left") or South. The Hawaiian Star Compass is oriented with North at the top.

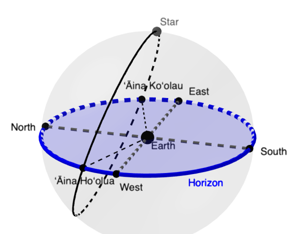
The star compass is divided into four quadrants, each named after winds in Hawaiʻi. *Koʻolau* is the Northeast quadrant named for the trade winds; *Hoʻolua* is the Northwest quadrant, *Kona* is the Southwest quadrant; and *Malanai* is the Southeast quadrant.

Each house on the star compass is given a name. The corresponding houses in the east and west share the same name. Starting from the east or west and moving northwards and southwards, the first house on either side of Hikina (East) and Komohana (West) is called *Lā* (Sun). It is followed by *ʻĀina* (Land), *Noio* (Tern), *Manu* (Bird), *Nālani* (Heavens), *Nā Leo* (Voices), and *Haka* (Empty). The 32 houses in the Hawaiian Star Compass correspond to the points of the 32-wind compass rose (Table 1.1.5).

Table 1.1.5 The houses of the Hawaiian Star Compass and the corresponding points on the 32-wind compass rose.

Star Compass	32 Point Compass		Star Compass	32 Point Compass	
House	Symbol	Name	House	Symbol	Name
Hikina	E	East	Komohana	W	West
Lā Koʻolau	EbN	East by North	Lā Kona	WbS	West by South
ʻĀina Koʻolau	ENE	East-northeast	ʻĀina Kona	WSW	West-southwest
Noio Koʻolau	NEbE	Northeast by East	Noio Kona	SWbW	Southwest by West
Manu Koʻolau	NE	Northeast	Manu Kona	SW	Southwest
Nālani Koʻolau	NEbN	Northeast by North	Nālani Kona	SWbS	Southwest by South
Nā Leo Koʻolau	NNE	North-northeast	Nā Leo Kona	SSW	South-southwest
Haka Koʻolau	NbE	North by east	Haka Kona	SbW	South by West
ʻĀkau	N	North	Hema	S	South
Haka Hoʻolua	NbW	North by West	Haka Malanai	SbE	South by East
Nā Leo Hoʻolua	NNW	North-northwest	Nā Leo Malanai	SSE	South-southeast
Nālani Hoʻolua	NWbN	Northwest by North	Nālani Malanai	SEbS	Southeast by South
Manu Hoʻolua	NW	Northwest	Manu Malanai	SE	Southeast
Noio Hoʻolua	NWbW	Northwest by West	Noio Malanai	SEbE	Southeast by East
ʻĀina Hoʻolua	WNW	West-northwest	ʻĀina Malanai	ESE	East-southeast
Lā Hoʻolua	WbN	West by North	Lā Malanai	EbS	East by South

Celestial bodies move along parallel paths across the sky from East to West, rising and setting in the same house, remaining in its hemisphere. For example, if a star arrives in the Koʻolau (northeastern) quadrant in the star house ʻĀina, it will arc overhead, staying in the northern hemisphere, and enter the horizon in the same house it arrived in, ʻĀina, but in the Hoʻolua (northwestern) quadrant (see Figure 1.1.6). Similarly, if a star arrives in the Malanai (southeastern) quadrant in the house Lā, it will remain in the southern hemisphere as it arcs overhead and enters the horizon in the house Lā in the Kona (southwestern) quadrant.



www.geogebra.org/material/iframe/id/nmjdg3kb

Figure 1.1.6 In the celestial sphere, stars rise in the east, arc across the sky, and set in the west. Each star will both rise and set in the same house.

The star compass also serves as a guide for determining direction based on wind and ocean swells. As the wind and swells move, they intersect the star compass diagonally. For example, if a wind blows from the house Noio in the Koʻolau (northeast) quadrant, it will blow in the direction of the Kona (southwest) quadrant and eventually exit in the same house, Noio.

Observations play a key role in determining direction using the star compass. At night, Thompson relies on approximately 220 stars, memorizing where they rise and set on the horizon to navigate. During the day, we can use the sun's position on the horizon to gauge direction, but this method is only effective when the sun is near the horizon at sunrise and sunset. Alternatively, one can memorize the wind and wave directions, checking for any changes between sunrise and sunset to establish their current direction.

The canoe itself can serve as a compass, as shown in Figure 1.1.7. From the navigator's seat on either corner of the stern (back) of the deck, you can observe features like the rising sun and mark its position on the Star Compass located on the canoe. It's essential to note that the locations of the houses on this Star Compass are in relation to the canoe, not to a fixed map. For instance, only when the canoe is pointed towards the north will Hikina (East) be the house to the right. Depending on the canoe's orientation, at other times, Hikina may appear several houses further up the deck.

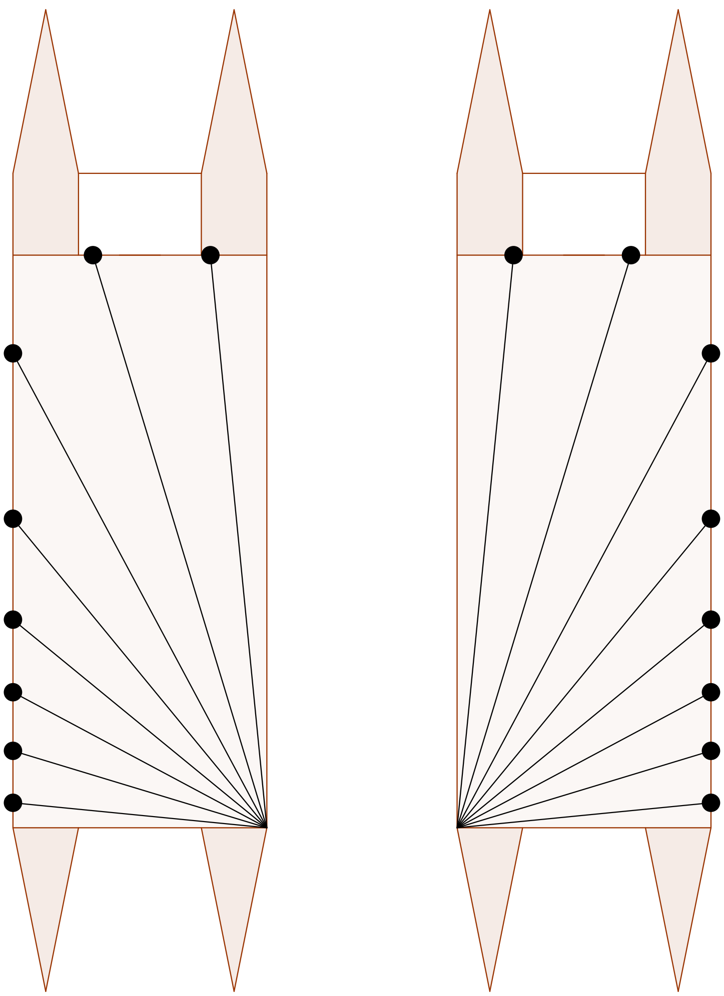


Figure 1.1.7 The deck of a canoe can be used as a compass to help crew and navigators.

1.1.3 Marshall Islands

The Marshallese people use stick charts as navigational tools. These stick charts are constructed using a lattice-like structure made from curved and straight sticks, typically formed by tying together the midribs of coconut fronds. The curved sticks represent the islands and how they bend and refract the ocean swells, while the straight sticks symbolize the major wave patterns in the surrounding waters.

The shells placed on the sticks indicate the relative locations of islands within the Marshall Islands archipelago. These shells serve as markers, helping navigators remember the positions of specific islands along their voyages.

Each stick chart is unique to its creator, reflecting their individual knowledge and experiences. The personalization of the stick charts allows navigators to develop a deep understanding of the ocean currents, wave patterns, and island locations in their specific region.

The stick charts serve as mental maps or navigational aids, allowing experienced navigators to visualize and recall the complex information while on their journeys. Navigators would memorize the stick charts, internalizing the knowledge embedded within them, enabling them to navigate the open ocean.

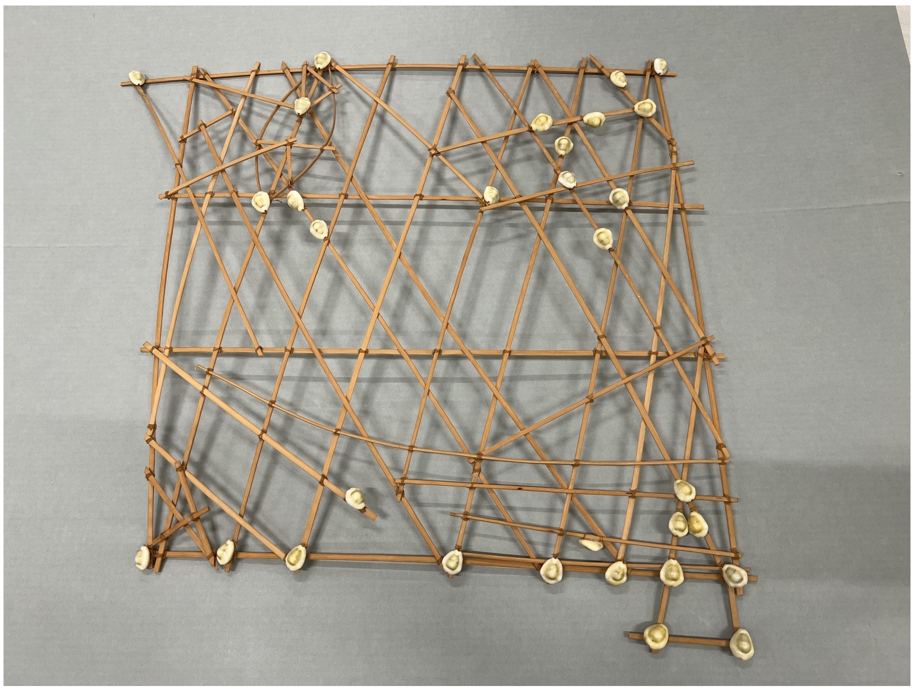
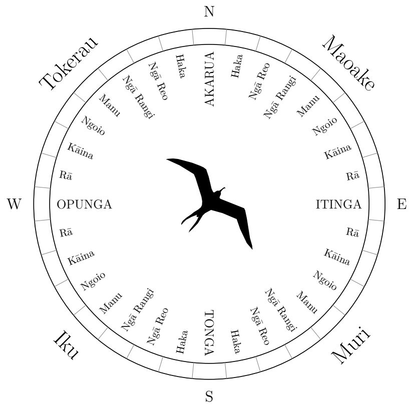


Figure 1.1.8 Marshallese stick chart.

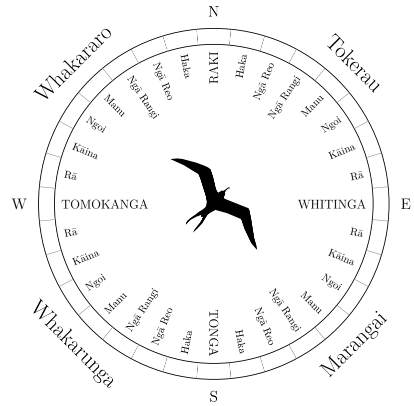
1.1.4 Elsewhere in the Pacific

In addition to the star compass, many cultures across the Pacific use a *wind compass*. Similar to the star compass, the wind compass is also a mental construct.

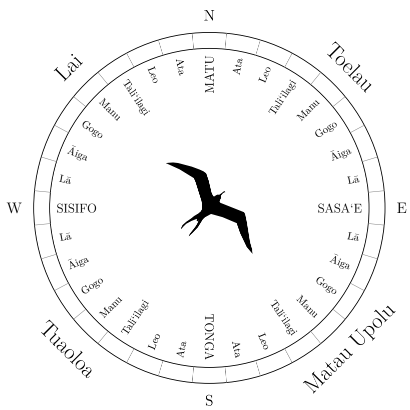
Other Pacific Island cultures have also adapted the modern Hawaiian Star Compass to their languages, as illustrated in Figure 1.1.9.



(a) Cook Islands.



(b) Māori.



(c) Sāmoan.

Figure 1.1.9 Examples of Star Compasses across the Pacific.

1.1.5 Exercises

1. Who developed the Hawaiian Star Compass?

Answer. Nainoa Thompson

2. The Hawaiian Star Compass was based on the Micronesian Star Compass, known as the *paafu*. Who shared the paafu with the Hawaiians?

Answer. Mau Piailug or Papa Mau

3. According to the Hawaiian Star Compass, what is the name for

(a) North

Answer. ʻĀkau

(b) East

Answer. Hikina

(c) South

Answer. Hema

(d) West

Answer. Komohana

4. What is the Hawaiian name for winds in the

(a) Northeast quadrant

Answer. Koʻolau

(b) Southeast quadrant

Answer. Malanai

(c) Southwest quadrant

Answer. Kona

(d) Northwest quadrant

Answer. Hoʻolua

Exercise Group. For each direction, identify the Hawaiian names of the corresponding house and quadrant in the Hawaiian Star Compass

5. Northwest by North

Answer. Nālani Hoʻolua

6. East-northeast

Answer. ʻĀina Koʻolau

7. North-northwest

Answer. Nā Leo Hoʻolua

8. Southeast

Answer. Manu Malanai

9. South by West

Answer. Haka Kona

10. East by South

Answer. Lā Malanai

11. Southwest by West

Answer. Noio Kona

12. Northeast by North

Answer. Nālani Koʻolau

13. South-southeast

Answer. Nā Leo Malanai

14. West-southwest

Answer. ʻĀina Kona

Exercise Group. Identify the corresponding point on the 32-wind compass for each house on the Hawaiian Star Compass

15. Nā Leo Kona

Answer. South-southwest

16. ʻĀina Malanai

Answer. East-southeast

17. ʻĀina Hoʻolua

Answer. West-northwest

18. Lā Koʻolau

Answer. East by North

19. Manu Koʻolau

Answer. Northeast

20. Haka Malanai

Answer. South by East

21. Nālani Kona

Answer. Southwest by South

22. Haka Hoʻolua

Answer. North by West

23. Noio Hoʻolua

Answer. Northwest by West

24. Noio Koʻolau

Answer. Northeast by East

25. The winter solstice in the southern hemisphere occurs around June 22. It is the time when the sun is at its lowest elevation in the sky, resulting in the shortest daylight of the year. During the winter solstice, the sun rises from its northernmost position, ʻĀina Koʻolau. In what house does the sun set during the winter solstice in the southern hemisphere?

Answer. ʻĀina Hoʻolua

26. The winter solstice in the northern hemisphere occurs around December 22 when the sun rises from its southernmost position, ʻĀina Malanai. In which house does the sun set during the winter solstice in the northern hemisphere?

Answer. ʻĀina Kona

27. Wind is coming from Nā Leo Kona. In what direction is the wind blowing?

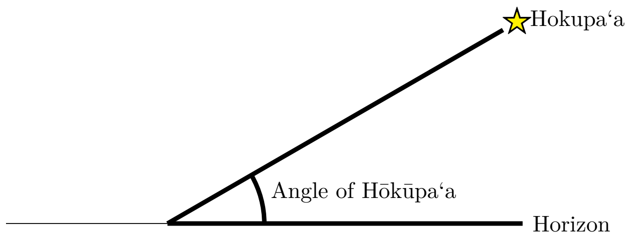
Answer. Nā Leo Koʻolau

28. Current is coming from Noio Hoʻolua. In what direction is the current heading?

Answer. Noio Malanai

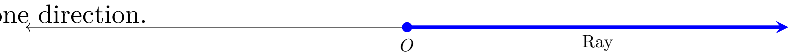
1.2 Angles and Their Measure

One method people use to identify their position is by looking at the *latitude*. These imaginary lines form circles around the Earth and run parallel to the Equator. The latitude of a place is defined as the angle between a line drawn from the center of the Earth to that point and the equatorial plane. For any point in the Northern Hemisphere, a navigator can measure their latitude by determining the angle that *Hōkūpaʻa* (also known as *Kūmau*, *Wuli wulifasmughet*, *Fuesemagut*, North Star, or Polaris) makes with the horizon.



During voyages, knowing the correct angles can make the difference between reaching your destination or missing it. Navigators carefully observe angles on the Hawaiian Star Compass to determine the entry and exit points of celestial bodies in the sky, as well as the direction of wind and current. In this section, we will explore the properties of angles and their measure.

Definition 1.2.1 A **ray** is a part of a line that begins at a point O and extends in one direction.



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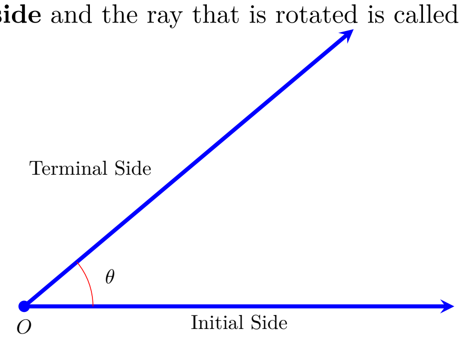
Definition 1.2.2 We can create an **angle**, θ, by rotating rays. First, we begin with two rays lying on top of each other and beginning at O. We let one ray be fixed and will rotate the second ray about the point O. The ray that is fixed is called the **initial side** and the ray that is rotated is called the **terminal side**.



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Remark 1.2.3 Angles are often measured using Greek letters. The commonly used Greek letters include θ, ϕ, α, β, and γ.

1.2.1 Degree

The **measure** of an angle is the amount of rotation from the initial side to the terminal side. One unit of measuring angles is the degree. One **degree**, denoted by $1°$, is $\frac{1}{360}$ of a complete circular revolution, so one full revolution is $360°$.

The Hawaiian Star Compass consists of 32 houses, each spanning $11.25°$ $(\frac{360°}{32})$. Assuming due East corresponds to $0°$ and the center of the House of

Hikina points due East, the border between Hikina and Lā Koʻolau will be half the angle of the house, 5.625° ($\frac{11.25°}{2}$). The angles for the other boundaries on the Hawaiian Star Compass are shown in Figure 1.2.4.

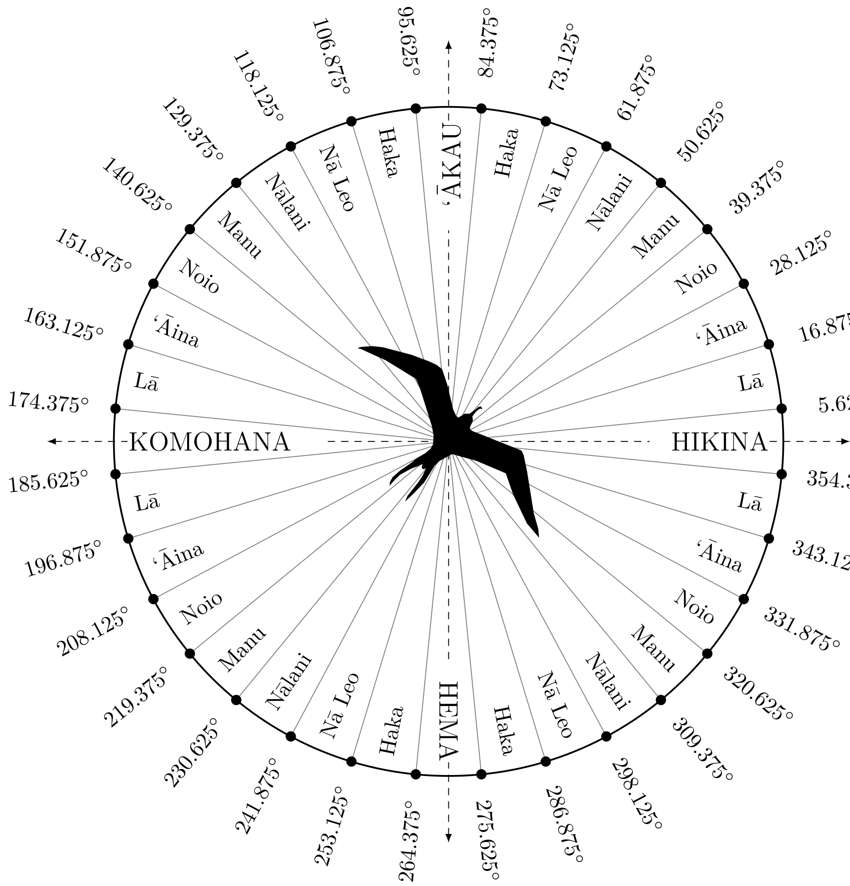


Figure 1.2.4 The Star Compass with the angles indicating the boundaries for each House.

Although decimals are commonly used to represent fractional parts of a degree, traditionally, degrees were represented in minutes and seconds. One **minute** or **arc minute**, denoted as $1'$, is equal to $\frac{1}{60}$ degrees, and one **second** or **arc second**, denoted as $1''$, is equal to $\frac{1}{60}$ minutes.

Remark 1.2.5 Conversion Between Degree, Minutes, and Seconds.

$$1° = 60' \qquad 1' = \left(\frac{1}{60}\right)^{\circ} \qquad 1'' = \left(\frac{1}{3600}\right)^{\circ}$$

$$1° = 3600'' \qquad 1' = 60'' \qquad 1'' = \left(\frac{1}{60}\right)'$$

Example 1.2.6 Convert angle from decimal degrees to degrees/ minutes/seconds. In the Star Compass (Figure 1.2.4), the angle between the houses Manu Hoʻolua (northwest) and Noio Hoʻolua (northwest by west) measures 140.625°. Represent this angle in degrees, minutes, and seconds.

Solution. First we will convert 0.625° to minutes using the conversion $1° = 60'$,

$$0.625° = 0.625° \cdot \frac{60'}{1°} = 37.5'$$

Since $1' = 60''$, we can convert $0.5'$ to seconds: $0.5' = 0.5' \cdot \dfrac{60''}{1'} = 30''$.

So $140.625° = 140°37'30''$. □

Example 1.2.7 Convert angle from degrees/minutes/seconds to decimal degrees. Convert $263°24'45''$ to decimal degrees.

Solution. We will first convert $24'$ and $45''$ to degrees.

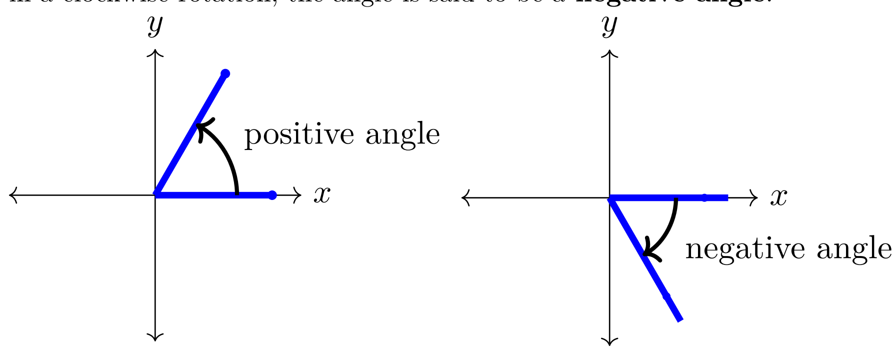
$$24' = 24 \cdot 1' = 24 \cdot \left(\frac{1}{60}\right)^{\circ} = 0.4°$$

and

$$45'' = 45 \cdot 1'' = 45 \cdot \left(\frac{1}{3600}\right)^{\circ} = 0.0125°$$

So $263°24'45'' = 263° + 24' + 45'' = 263° + 0.4° + 0.0125 = 263.4125°$ □

Definition 1.2.8 If an angle is drawn on the xy-plane, and the vertex is at the origin, and the initial side is on the positive x-axis, then that angle is said to be in **standard position**. If the angle is measured in a counterclockwise rotation, the angle is said to be a **positive angle**, and if the angle is measured in a clockwise rotation, the angle is said to be a **negative angle**.



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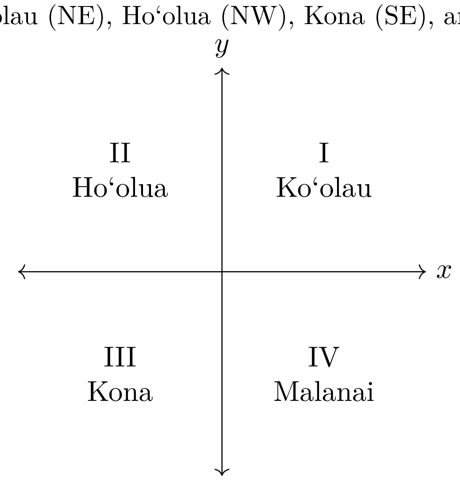
Definition 1.2.9 When an angle is in standard position, the terminal side will either lie in a quadrant or it will lie on the x-axis or y-axis. An angle is called a **quadrantal angle** if the terminal side lies on x-axis or y-axis. The two axes divide the plane into four **quadrants**. In the Cartesian plane, the four quadrants are Quadrant I, II, III, and IV. The corresponding quadrants of Star Compass are Koʻolau (NE), Hoʻolua (NW), Kona (SE), and Malanai (SW).



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Definition 1.2.10 Coterminal angles are angles in standard position that have the same initial side and the same terminal side. Any angle has infinitely many coterminal angles because each time we add or subtract $360°$ from it, the resulting angle has the same terminal side. ◊

Example 1.2.11 Coterminal angles. $90°$ and $450°$ are coterminal angles since $450° - 360° = 90°$. □

To determine the quadrant in which an angle lies, add or subtract one revolution ($360°$) until you obtain a coterminal angle between $0°$ and $360°$. The quadrant where the terminal side lies is the quadrant of the angle. Quadrantal angles do not lie in any quadrant.

Example 1.2.12 Determine the corresponding house and quadrant in the Star Compass. Determine the quadrant in which each angle lies and name the corresponding House and quadrant in the Star Compass (Figure 1.2.4).

1. $140°$
2. $-770°$
3. $923°$

Solution.

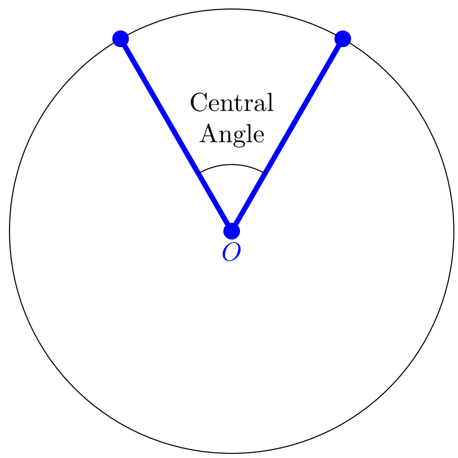
1. Since $90° < 140° < 180°$, $140°$ lies in Quadrant II, or Manu Hoʻolua.
2. Since $-770° < 0°$, we first add 3×360 to $-770°$ to obtain an angle $0°$ and $360°$,
$$-770° + 3 \times 360° = 310°$$
So $310°$ and $-770°$ are coterminal. Since $270° < 310° < 360°$, $310°$ lies in Quadrant IV, or Manu Malanai.
3. Since $923° > 360°$ we begin by subtracting $2 \times 360°$
$$923° - 2 \times 360° = 203°$$
So $203°$ and $923°$ are coterminal. Since $180° < 203° < 270°$, $923°$ lies in Quadrant III or ʻĀina Kona.

□

Example 1.2.13 Determine the corresponding quadrant given its location in the Star Compass. What is the corresponding quadrant for Nālani Kona?

Solution. Locating Nālani Kona in the Star Compass, we see it is in Quadrant III. □

Definition 1.2.14 A **central angle** is a positive angle formed at the center of a circle by two radii.



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Remark 1.2.15 Heading and Azimuth. In navigation, the direction a wa‘a is pointed towards is referred to as the **heading**. Unlike in trigonometry, where it is conventional to define an angle in standard position, i.e., 0° lies along the positive x-axis, in navigation, North corresponds to a heading of 0° and positive angles are measured in a clockwise rotation (see Figure 1.2.16).

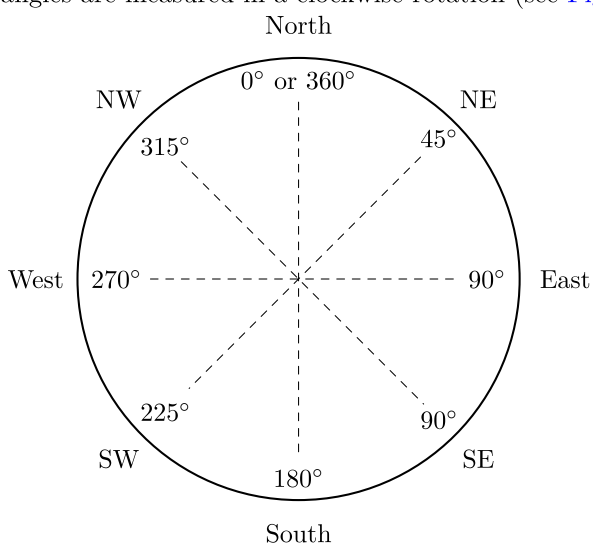


Figure 1.2.16 The cardinal directions for headings are as follows: 0° (or 360°) points north, 90° points east, 180° points south, and 270° points west.

The Star Compass can now be presented in terms of heading angles, as demonstrated in Figure 1.2.17.

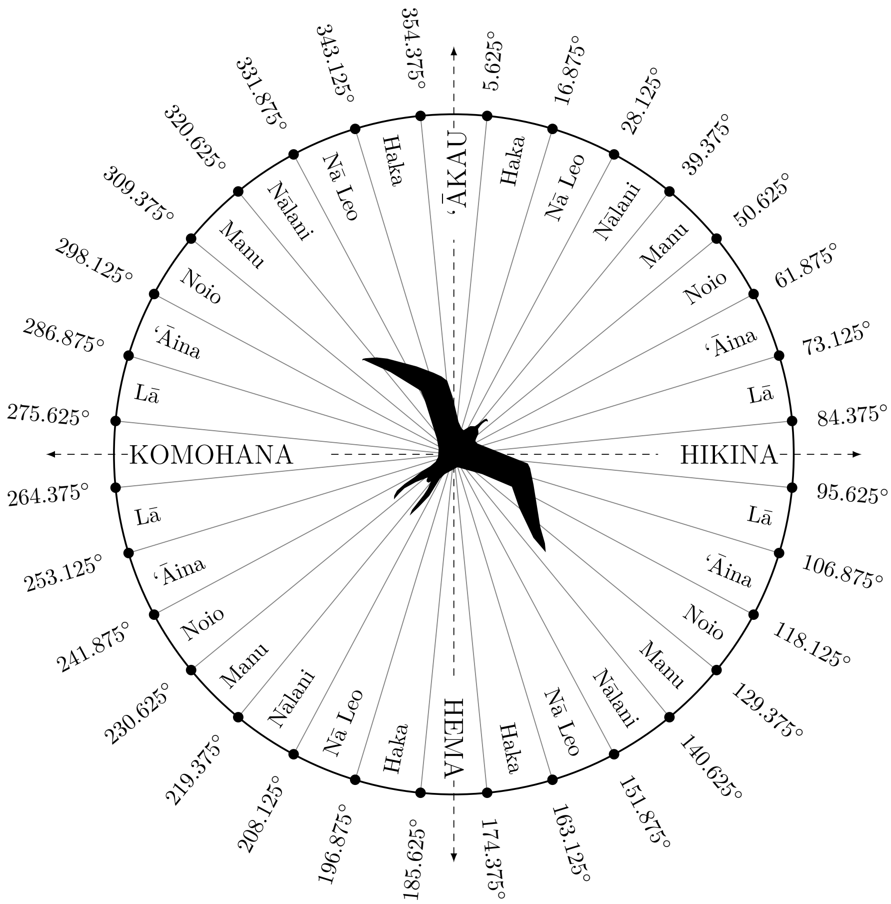


Figure 1.2.17 The Star Compass is presented in terms of heading, with the angles indicating the boundaries for each House.

In astronomy and navigation, the position of a celestial body as it rises or sets on the horizon can be measured using the **azimuth**, which indicates the direction of celestial objects relative to an observer's position. Similar to heading, azimuth starts from the north and increases clockwise.

In navigation, "heading" typically denotes the direction an object like a canoe or wind is pointed, whereas "azimuth" pertains to the angular measurement of celestial bodies on the horizontal plane. Both "heading" and "azimuth" measure angles in degrees, beginning from north and progressing clockwise. Unless specified otherwise to use the heading or azimuth angle (in which case, refer to Figure 1.2.17), this book will use Figure 1.2.4 for the angles of the Star Compass.

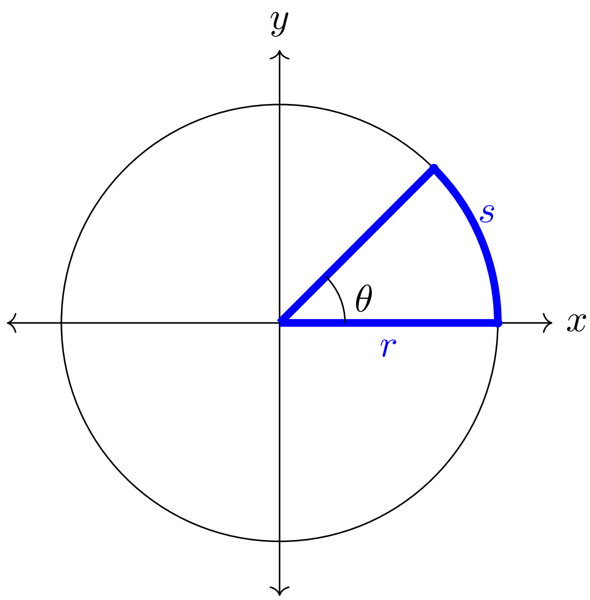
1.2.2 Radian

Another way to measure an angle is with *radians*, which measure the the arc of a circle that is formed from an angle.

Definition 1.2.18 Definition of a Radian. The **radian measure** of a central angle in a circle is the ratio of the length of the arc on a circle subtended by the angle to the radius. If r is the radius of the circle, θ is the angle, and s is the arc length, then we have the following

$$\theta = \frac{s}{r}$$

A radian is abbreviated by **rad**.



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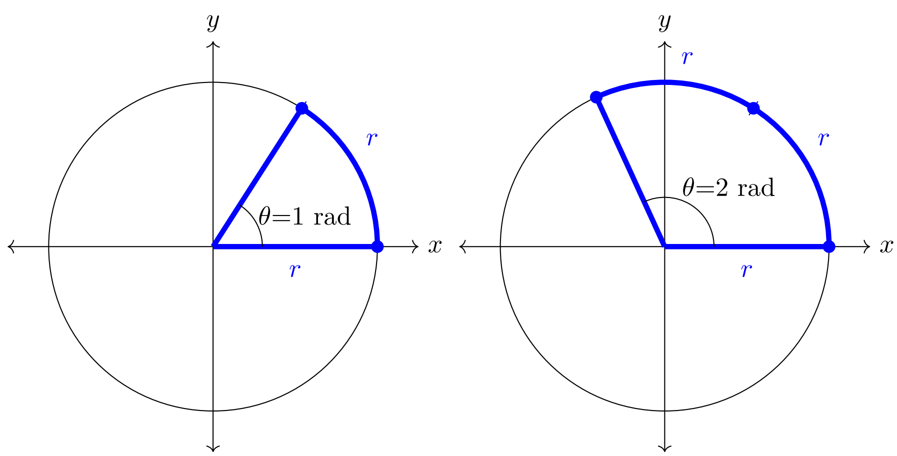
The measure of a central angle obtained when the length of the arc is also equal to the radius, r, is called *one radian* (1 rad). Similarly, if $\theta = 2$ rad, then the arc length equals $2r$.



The circumference of a circle is $C = 2\pi r$. This means that the circumference is $2\pi \approx 6.28$ times the radius. Consequently, if we were to use a piece of string with the length of the radius, we would need six pieces of string plus a fractional piece of the string, as shown in Figure 1.2.19.

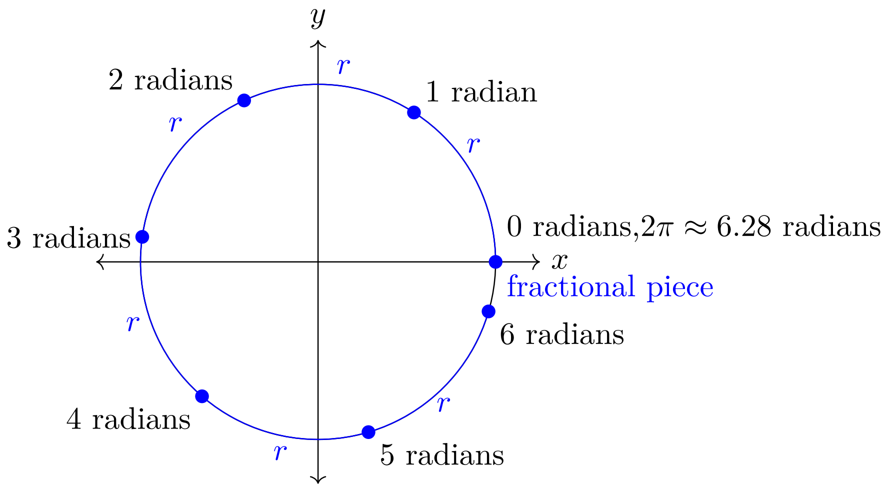


Figure 1.2.19 One rotation of the unit circle is $2\pi \approx 6.28$ radians.

Remark 1.2.20 Relationships Between Degrees and Radians. If a circle with radius 1 is drawn, it has $360°$, and the full arc length is the circumference, which is 2π. Therefore,the relationship between degrees and radians is:

$$360° = 2\pi \text{ radian, or } 180° = \pi\text{radian}$$

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$1° = \frac{\pi}{180} \text{ radian}$$

Remark 1.2.21 Converting Between Degrees and Radians.

1. To convert degree to radians, multiply by $\dfrac{2\pi \text{ radians}}{360°}$ or $\dfrac{\pi \text{ radian}}{180°}$
2. To convert radians to degrees, multiply by $\dfrac{360°}{2\pi\text{radians}}$ or $\dfrac{180°}{\pi \text{ radian}}$

Example 1.2.22 Express $45°$ in radians.

Solution. $45° = 45° \left(\dfrac{2\pi \text{ radians}}{360°}\right) = \dfrac{\pi}{4}\text{radians}$ □

Example 1.2.23 Express $\dfrac{5\pi}{6}$ in degrees.

Solution. $\dfrac{5\pi}{6}\text{rad} = \dfrac{5\pi}{6}\text{rad}\left(\dfrac{360°}{2\pi \text{ rad}}\right) = 150°$ □

Using this method, we can obtain Table 1.2.24 of common angles used in trigonometry and the corresponding radian and degree measures.

Table 1.2.24 Commonly Used Angles in Trigonometry: Degrees and Radians

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
Radians	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degrees	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$

1.2.3 Arc Length

Recall that the definition of a radian is the ratio of the arc length to the radius of a circle, $\theta = \frac{s}{r}$. By rearranging this formula, we can obtain a formula for the arc length of a circle.

Theorem 1.2.25 *In a circle of radius r, the* arc length*, s, subtended by a central angle (in radians), θ, is*

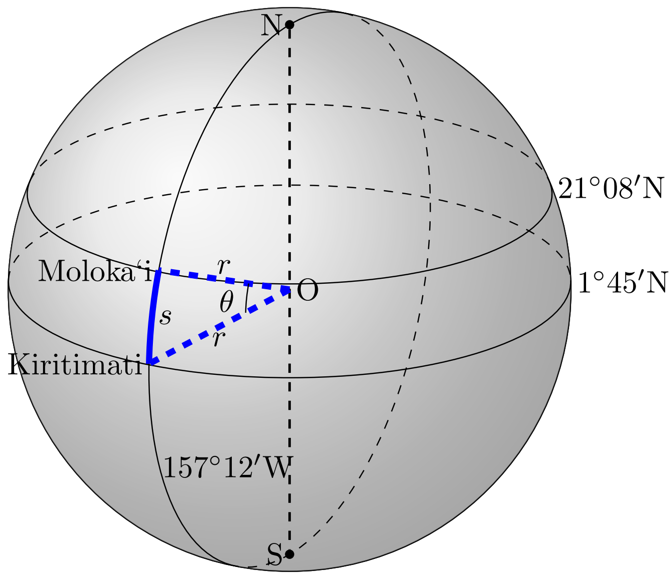
$$s = r\theta$$

If θ is given in degrees, then $s = 2\pi r \cdot \left(\frac{\theta}{360°}\right)$.

Example 1.2.26 Find the length of an arc of a circle with radius 10 cm subtended by an angle of 2 radians.

Solution. Using the arc formula we get $s = 10\text{cm} \cdot 2\text{rad} = 20\text{cm}$. □

Example 1.2.27 Kiritimati, also known as Christmas Island, is an atoll in the Republic of Kiribati. Kiritimati's location west of the International Date Line makes it one of the first places in the world to welcome the New Year, while Hawaiʻi is one of the last places. Although Kiritimati and Molokaʻi share the same longitude at $157°12'$ west (meaning Molokaʻi is directly north of Kiritimati), both islands are 24 hours apart. For example, if the time on Oʻahu is 3:00 pm on Thursday, then at that same moment it is 3:00 pm on Friday in Kiritimati. Find the distance between Kiritimati ($1°45'$ north latitude) and Molokaʻi ($21°08'$ north latitude). Assume the radius of Earth is 3,960 miles and that the central angle between the two islands is the difference in their laititudes.



Solution. The measure of the central angle between the two islands is

$$\begin{aligned}\theta &= 21°08' - 1°45' \\ &= 19°23' \\ &= 19° + \frac{23}{60}^{\circ} \\ &\approx 19.3833°\end{aligned}$$

To find the distance, we use Theorem 1.2.25 to find the arc length:

$$s = 2\pi r \cdot \left(\frac{\theta}{360°}\right) \approx 2\pi \cdot (3,960 \text{ miles})\frac{19.3833°}{360°} \approx 1,340 \text{ miles}$$

So the distance between Kiritimati and Moloka'i is approximately 1,340 miles. □

1.2.4 Area of a Sector of a Circle

Definition 1.2.28 Area of a Sector. The **area of the sector** of a circle of radius r formed by a central angle of θ is

$$\begin{aligned}A &= \frac{\theta}{360°} \cdot \pi \cdot r^2, \text{ when } \theta \text{ is in degrees} \\ A &= \frac{1}{2} r^2 \theta, \text{ when } \theta \text{ is in radians}\end{aligned}$$

◊

Notice the ratio $\dfrac{\theta}{360°}$ is the proportion of the angle θ (in degrees) to one complete circle. Additionally, the circumference of a circle is given by $2\pi r$. Therefore, the arc length is simply the proportion of the central angle to the whole circle multiplied by the circumference of the circle.

$$s = \text{arc length} = (\text{proportion of circle}) \cdot (\text{circumference}) = \left(\frac{\theta}{360°}\right) \cdot (2\pi r)$$

Similarly, the area of a circle is given by πr^2. So the area of sector is the proportion of the central angle to the whole circle multiplied by the area of the circle.

$$A = \text{area of sector} = (\text{proportion of circle}) \cdot (\text{area of circle}) = \left(\frac{\theta}{360°}\right) \cdot \left(\pi r^2\right)$$

Theorem 1.2.29 *Given a circle of radius r formed by a central angle of θ, then the arc length and area of the sector formed by θ can be expressed as the proportion of the angle to the full circle multiplied by the circumference and area of the circle, respectively.*

$$s = \textit{(proportion of circle)} \cdot \textit{(circumference)} = \left(\frac{\theta}{360°}\right) \cdot (2\pi r)$$

and

$$A = \textit{(proportion of circle)} \cdot \textit{(area of circle)} = \left(\frac{\theta}{360°}\right) \cdot \left(\pi r^2\right)$$

Example 1.2.30 When sailing, Hōkūleʻa cannot make headway by sailing directly into the wind. It can only sail beyond 67° in either direction from the wind (Figure 1.2.31). If Hōkūleʻa sails for 50 miles, what is the area of the sector that cannot be sailed? Round your answer to the nearest square mile.

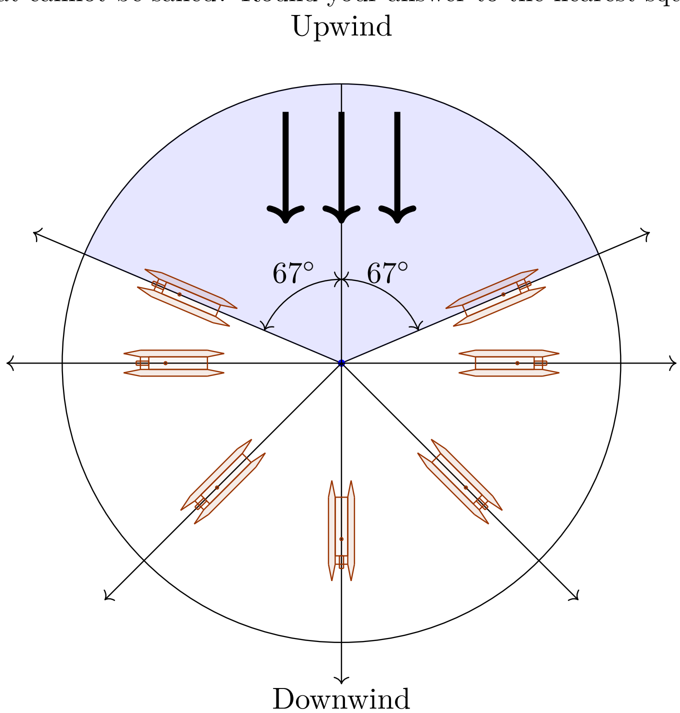


Figure 1.2.31 Hōkūleʻa cannot sail within 67° into the direction of the wind.

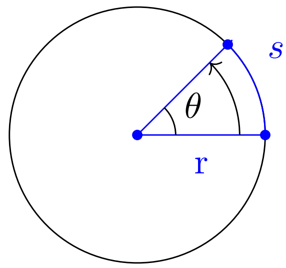
Solution. The angle is $\theta = 2 \cdot 67° = 134°$ and the radius is $r = 50$ miles. So the area is given by

$$A = \frac{\theta}{360°}\pi \cdot r^2 = \frac{134}{360}\pi \cdot 50^2 \approx 2,923 \text{ square miles}$$

□

1.2.5 Angular Velocity and Linear Speed

Consider an object moving along a circle as shown below. There are two ways to describe the circular motion of this object: *linear speed* which measures the distance traveled; and *angular speed* which measures the rate at which the central angle changes.



Definition 1.2.32 Linear Speed. Suppose an object moves along a circle with radius r and θ (measured in radians) is the angle transversed in time t. Let s be the distance the object traveled in time t. Then the **linear speed**, v, of the object is given by

$$v = \frac{s}{t}$$

◊

Definition 1.2.33 Angular Speed. Suppose an object moves along a circle. Let θ (measured in radians) be the angle transversed by the object in time t. The **angular speed**, ω, of the object is given by

$$\omega = \frac{\theta}{t}$$

◊

Notice that we can rearrange the angular speed to get $\theta = \omega t$. Since s is an arc length, we have $s = r\theta$, and thus we can write the linear speed as

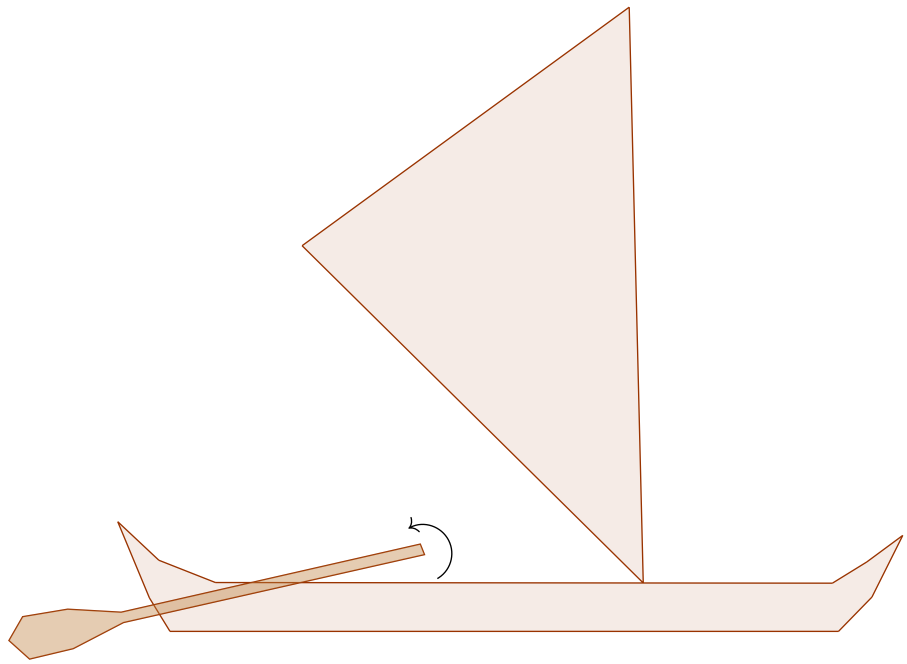
$$v = \frac{s}{t} = \frac{r\theta}{t} = \frac{r\omega t}{t} = r\omega$$

Definition 1.2.34 Linear Speed. Suppose an object moves along a circle with radius r and an angular speed ω (measured in radians per unit time). Then the **linear speed**, v, of the object is given by

$$v = r\omega$$

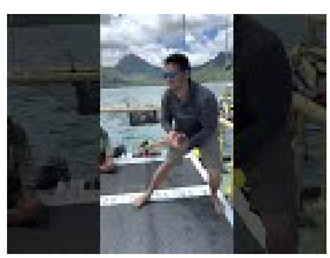
◊

Example 1.2.35 Une.



One method a wa‘a uses to change direction is with the *hoe uli*, or the steering paddle. When a sharp turn is needed for maneuvers such as tacking, the steersperson will turn the handle of the hoe uli in a circular motion, as a lever to scoop the paddle in the water and change the heading of a vessel. This move called *une* (prounced oo-NAY although it is often mispronounced as oo-NEE) literally translates to "lever." If the steerperson is performing an une at a rate of 25 rotations per minute and the radius of the circular movement is 2 feet, calculate

1. the angular speed measured in radians per minute
2. the linear speed of the hoe uli in miles per hour (round your answer to two decimal places)



YouTube: `https://www.youtube.com/watch?v=1A2UDMDE3Uc`

Figure 1.2.36 A wa‘a (canoe) can change directions by rotating the hoe uli (steering sweep) in a process known as une.

Solution.

1. We are given the angular speed is $\omega = 25$ revolutions per minute. To convert our angular speed to radians per minute, we use the fact that one revolution is 2π radians to get

$$\omega = 25\frac{\text{revolution}}{\text{minute}} = 25\frac{\text{rev}}{\text{minute}} \cdot 2\pi\frac{\text{radians}}{\text{revolution}} = 50\pi\frac{\text{radians}}{\text{minute}}$$

Thus the hoe uli is moving at an angular speed of 50π radians per second.

2. Since the radius is $r = 2$ ft and the angular speed is 50π radians per minute, we can use Definition 1.2.34 to calculate the linear speed

$$v = r\omega = 2\text{ft} \cdot 50\pi \frac{\text{rad}}{\text{min}} \cdot \frac{\text{mile}}{5280\text{ft}} \cdot \frac{60\text{min}}{\text{hr}} \approx 3.57 \frac{\text{miles}}{\text{hour}}$$

Thus the steersperson is moving the hoe uli at a linear speed of 3.57 mph.

□

1.2.6 Exercises

Exercise Group. Given an angle, θ, identify the house and quadrant on the Hawaiian Star Compass.

1. $\theta = 336°$

Answer. ʻĀina Malanai

2. $\theta = 240°$

Answer. Nālani Kona

3. $\theta = 35°$

Answer. Noio Koʻolau

4. $\theta = 221°$

Answer. Manu Kona

5. $\theta = 108°$

Answer. Nā Leo Hoʻolua

6. $\theta = 190°$

Answer. Lā Kona

7. $\theta = 323°$

Answer. Noio Malanai

8. $\theta = 158°$

Answer. ʻĀina Hoʻolua

9. $\theta = 172°$

Answer. Lā Hoʻolua

Exercise Group. Convert the given angle θ to a decimal in degrees rounded to two decimal places.

10. $\theta = 20°50'30''$

Answer. 20.84°

11. $\theta = 80°25'16''$

Answer. 80.42°

12. $\theta = 7°22'38''$

Answer. 7.38°

13. $\theta = 330°14'12''$

Answer. 330.24°

14. $\theta = 168°22'26''$

Answer. 168.37°

15. $\theta = 49°27'12''$

Answer. 49.45°

16. $\theta = 327°38'58''$

Answer. 327.65°

17. $\theta = 281°48'50''$

Answer. 281.81°

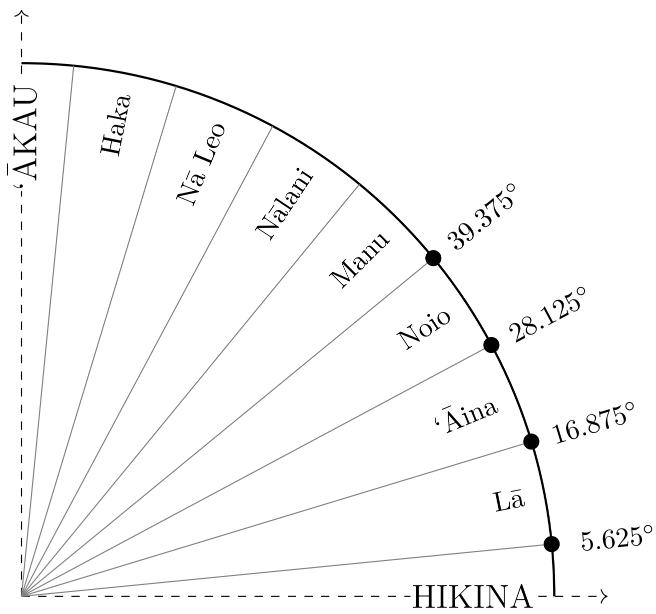
18. $\theta = 134°53'47''$

Answer. 134.90°

19. Sirius, the brightest star in the night sky, has been known by various names in different cultures and languages around the world. In Tahiti, it is called Taurere and is considered a zenith star as it passes directly overhead. Taurere has a **declination** of $-16°42'58''$, representing its angular distance south of the celestial equator. Express Taurere's declination as a decimal rounded to two decimal places.

Answer. 16.72°

Exercise Group. Recall the Star Compass with the boundaries for each House. Write the angles for the boundaries between the following houses in the Koʻolau quadrant in terms of degrees, minutes, and seconds.



20. Between Hikina and Lā (5.625°)

Answer. 5°37′30″

21. Between Lā and ʻĀina (16.875°)

Answer. 16°52′30″

22. Between ʻĀina and Noio (28.175°)

Answer. 28°7′30″

23. Between Noio and Manu (39.375°)

Answer. 39°22′30″

Exercise Group. Convert the given angle θ to degrees/minutes/seconds rounded to the nearest second and identify the house and quadrant on the Hawaiian Star Compass.

24. $\theta = 258.39°$

Answer. $\theta = 258°23'28''$ Haka Malanai

25. $\theta = 212.43°$

Answer. $\theta = 212°25'33''$ Noio Kona

26. $\theta = 244.97°$

Answer. $\theta = 244°57'59''$ Nā Leo Kona

27. $\theta = 93.95°$

Answer. $\theta = 93°57'6''$ Haka Hoʻolua

28. $\theta = 135.625°$

Answer. $\theta = 135°37'30''$ Manu Hoʻolua

29. $\theta = 162.52°$

Answer. $\theta = 162°59'56''$ Manu Hoʻolua

30. $\theta = 328.21°$

Answer. $\theta = 328°12'9''$ Noio Malanai

31. $\theta = 48.12°$

Answer. $\theta = 48°7'17''$ Manu Koʻolau

32. $\theta = 241.27°$

Answer. $\theta = 241°16'15''$ Nā Lani Kona

Exercise Group. Convert the given angle θ to radians and identify the house and quadrant on the Hawaiian Star Compass. Keep your answers in terms of π.

33. $\theta = 102°$

Answer. $\theta = \frac{17\pi}{30}$ Haka Ho‘olua

34. $\theta = 96°$

Answer. $\theta = \frac{8\pi}{15}$ ‘Ākau

35. $\theta = 190°$

Answer. $\theta = \frac{19\pi}{18}$ Lā Kona

36. $\theta = 122°$

Answer. $\theta = \frac{61\pi}{90}$ Nālani Ho‘olua

37. $\theta = 32°$

Answer. $\theta = \frac{8\pi}{45}$ Noio Ko‘olau

38. $\theta = 236°$

Answer. $\theta = \frac{59\pi}{45}$ Nālani Kona

39. $\theta = 82°$

Answer. $\theta = \frac{41\pi}{90}$ Haka Ko‘olau

40. $\theta = 228°$

Answer. $\theta = \frac{19\pi}{15}$ Manu Kona

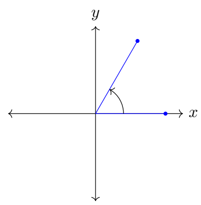
41. $\theta = 332°$

Answer. $\theta = \frac{83\pi}{45}$ ‘Āina Ko‘olau

Exercise Group. Draw the angle in standard position.

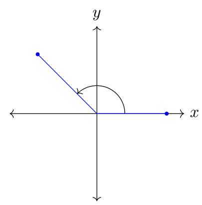
42. $60°$

Answer.



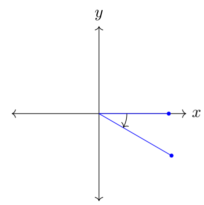
43. $135°$

Answer.



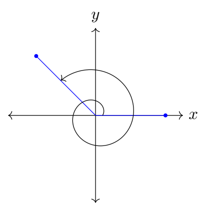
44. $-30°$

Answer.



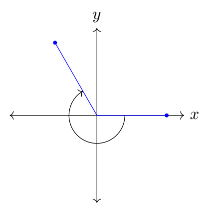
45. $495°$

Answer.



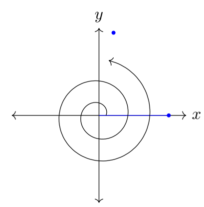
46. $-240°$

Answer.



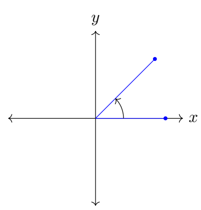
47. $800°$

Answer.



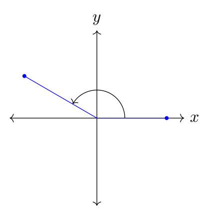
48. $\frac{\pi}{4}$

Answer.



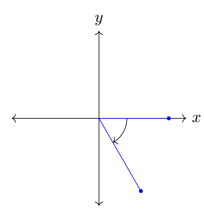
49. $\frac{5\pi}{6}$

Answer.



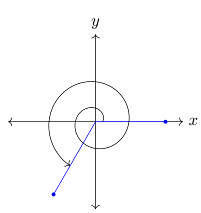
50. $-\frac{\pi}{3}$

Answer.



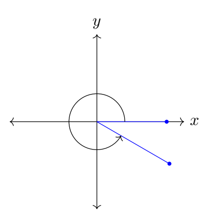
51. $\frac{10\pi}{3}$

Answer.



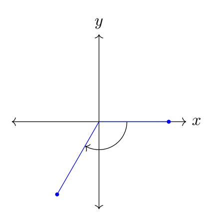
52. $\frac{11\pi}{6}$

Answer.



53. $-\frac{2\pi}{3}$

Answer.



Exercise Group. Convert the given angle from degrees to radians. Round your answer to two decimal places.

54. $27°$

Answer. 0.47

55. $63°$

Answer. 1.10

56. $-39°$

Answer. -0.68

57. $200°$

Answer. 3.49

58. $415°$

Answer. 7.24

59. $105°$

Answer. 1.83

Exercise Group. Convert the given angle from radians to degrees. Round your answer to two decimal places.

60. $\frac{3\pi}{5}$

Answer. $108°$

61. $\frac{8\pi}{3}$

Answer. $480°$

62. $-\frac{4\pi}{7}$

Answer. $-102.86°$

63. $\frac{15\pi}{4}$

Answer. $675°$

64. -1.5

Answer. $-85.94°$

65. 3

Answer. $171.89°$

Exercise Group. Determine whether the two given angles in standard position are coterminal.

66. $50°$, $410°$

Answer. Yes

67. $-60°$, $330°$

Answer. Yes

68. $30°$, $1110°$

Answer. No

69. $-210°$, $150°$

Answer. Yes

70. $-807°$, $93°$

Answer. No

71. $-757°$, $683°$

Answer. Yes

Exercise Group. Find an angle between $0°$ and $360°$ that is coterminal with the given angle

72. $405°$

Answer. $45°$

73. $600°$

Answer. $240°$

74. $1035°$

Answer. $315°$

75. $-300°$

Answer. $60°$

76. $381°$

Answer. $21°$

77. $-754°$

Answer. $326°$

Exercise Group. Given a circle with radius r, calculate (a) the length of the arc subtended by a central angle θ; and (b) the area of a sector with central angle θ. Round your answer to four decimal places.

78. $r = 10$ in, $\theta = 45°$

Answer.

(a) 7.8540 in;

(b) 39.2699 in^2

79. $r = 5$ m, $\theta = 120°$

Answer.

(a) 2.0944 m;

(b) 26.1799 m^2

80. $r = 3$ mi, $\theta = \frac{\pi}{3}$ radians

Answer.

(a) 3.1416 mi;

(b) 4.7124 mi^2

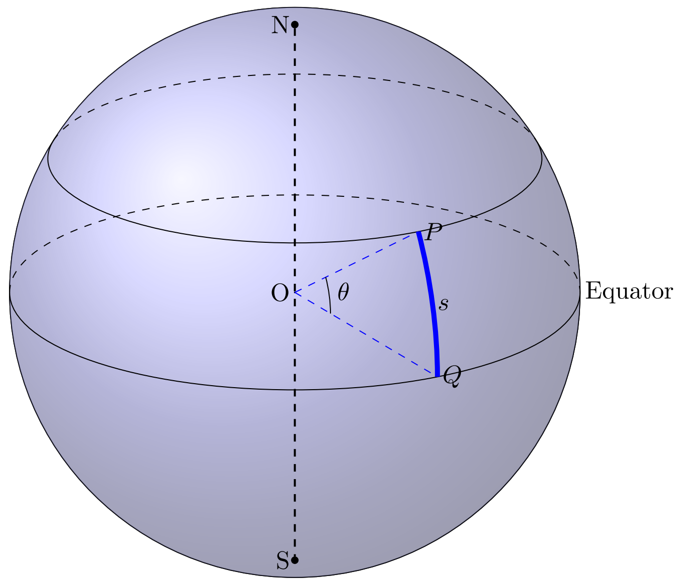
81. $r = 8$ cm, $\theta = 2$ radians

Answer.

(a) 16 cm;

(b) 64 cm^2

Exercise Group. At the start of this section, you learned that the **latitude** of a place is the angle between a line drawn from the center of the earth to that point and the equatorial plane. If the radius of the Earth is 3,959 miles, calculate the arc length, s, along the surface of the earth for each value of θ:



82. $\theta = 1^\circ$ of latitude (in miles, rounded to 2 decimals)

Answer. 69.10 miles

83. $\theta = 1'$ of latitude (in miles, rounded to 2 decimals). A *nautical mile*, frequently used in navigation, is slightly longer than a mile on land. One nautical mile was historically defined to be the arc length corresponding to one minute of latitude. Check your answer with the value of one nautical mile.

Answer. 1.15 miles

84. $\theta = 1''$ of latitude (in feet, rounded to the nearest integer).

Answer. 101 feet

85. The *oeoe*, or Hawaiian bullroarer, is made by drilling holes into a kamani seed or coconut shell, then threading a long string through the holes to secure it. When the oeoe is swung by the string, a whistling sound is produced, similar to the sound of the wind on the top of mountains. If a girl is swinging an oeoe at the end of 3 foot long rope at a rate of 180 revolutions per minute, calculate

(a) The angular speed measured in radians per minute.

Answer. 360π radians/minute

(b) The linear speed of the shell in miles per hour (round to two decimal places).

Answer. 38.56 miles/hour

86. Earth completes one rotation around the Sun approximately every 365.25 days. We will assume the orbit is a circle, and that the Earth is 92.9 million miles from the Sun.

(a) How far does the Earth travel in one day, expressed as millions of miles?

Hint. First determine the angle or proportion of a rotation that Earth travels in one day, then calculate the arc length of Earth's orbit.

Answer. 1.6 million miles

(b) How for does the Earth travel in 30 days, expressed as millions of miles?

Answer. 47.9 million miles

(c) How far does the Earth travel in one rotation around the sun, expressed as millions of miles?

Answer. 583.7 million miles

(d) What is the linear speed of Earth as it orbits the Sun? Express your answer in miles per hour.

Answer. $66,588$ miles per hour

87. As the the moon orbits the Earth, different parts of its surface become illuminated by the Sun which we call moon phases. The moon completes one rotation about Earth in approximately 27.3 days. If we assume its orbit is circular and the moon is 239,000 miles from Earth, calculate the linear speed of the moon, expressed as miles per hour.

Answer. 2,292 miles per hour

88. At 17.7° S latitude, the city of Nadi, Fiji is $6,071$ km from the Earth's axis of rotation. In 24 hours, Nadi will have traveled one rotation around Earth or $2\pi \cdot (6,071)$ km. The city of Port Vila, Vanuatu lies 967 km directly to the west of Nadi, Fiji. As the Earth rotates, how many minutes sooner will the people of Nadi see the Sun rise than the people in Port Vila, rounded to one decimal?

Hint. The proportion of distance between the two cities to the distance traveled in one rotation is the same as the proportion of the time it takes to see the sun between the two cities to time it takes to complete one rotation.

Answer. 36.5 minutes

89. In Example 1.2.30, we learned that wa‘a cannot sail directly into the wind. For each of the following wa‘a and distance traveled, determine the area of the sector that cannot be sailed? Recall that 1 house $= 11.25°$. Round your answer to the nearest square mile.

(a) Makali‘i sails for 25 miles and cannot sail within 4 houses from the direction of the wind.

Answer. 491 square miles

(b) Alingano Maisu sails for 15 miles and cannot sail within 3 houses from the direction of the wind.

Answer. 133 square miles

90. Navigating by the Sun: Using Solar Declination and Rising Sun to Orient on a Canoe. The position of the rising or setting sun changes throughout the year. **Solar declination** (denoted as δ) is the angle between the direction where the Sun rises (or sets) and due east (or due west) on the horizon. It represents how far north or south the Sun is from the celestial equator, projected onto the Earth's equatorial plane. Solar declinations to the north are positive, while those to the south are negative. At the Equinoxes (around March 20th and September 22nd), the solar declination is 0° ($\delta = 0°$), as the Sun is directly above the equator. During the December solstice, around December 22, the Sun rises from its most southern position, 23.5 degrees south of due east ($\delta = -23.5°$), and during the June solstice, around June 22, the Sun rises from its most northern position, 23.5 degrees north of due east ($\delta = 23.5°$).

A navigator can use their knowledge of the rising sun to help orient themselves. For example, on May 22, the solar declination is $\delta = 20°16'$. If the navigator identifies where the Sun rises on the equinox, she can measure $20°16'$ south to identify East and can then orient herself accordingly.

(a) What is the azimuth of the sun?

Answer. $69°44'$

(b) What house is the sun rising in?

Answer. ʻĀina Koʻolau

(c) What house does the sun set in?

Hint. Celestial bodies rise and set in the same house but different quadrants.

Answer. ʻĀina Hoʻolua

(d) If the canoe is sailing with the rising sun on the port side (left) and the navigator measures the angle between the direction of the canoe and the sun as being 90°, what is the heading of the canoe?

Answer. $159°44'$

(e) What house is the canoe sailing in?

Answer. Nā Leo Malanai

91. Swells are one of the most consistent navigational tools used to keep on a course because they can remain constant over time. On 27 May 2023, while sailing on the vaka Paikea from Rarotonga to Apia, you finished your shift and are ready to take a nap. Before you lay down, you take note that the canoe has a heading of 310° and the swells are coming from the southwest (Manu Kona) and hitting the canoe at 5° above directly left of the canoe. When you wake, you noticed the swells are now hitting the canoe from 25° to the left of your heading. You are aware that the swells couldn't have changed this fast and conclude that while you were asleep, the canoe changed its heading. Assuming the swell was constant, determine your new heading.

Hint.

(a) Start by drawing a diagram that represents the heading of the canoe before the nap. Mark the initial heading as °>.

(b) Next, draw the direction of the swells with respect to the canoe on the same diagram. The swells are coming from the southeast (Manu Kona) and hitting the canoe at °> above straight from the left of the canoe.

(c) Now, draw another diagram of the swells and the canoe, but this time, represent the swells hitting the canoe from °> to the left of straight in front (Nā Leo Hoʻolua).

(d) Observe that the swells couldn't have changed direction so fast while you were asleep. Thus, the change in the direction of the swells must be due to the canoe changing its heading.

(e) Use the relationship between the angle of the swell and the angle between the swell and the canoe to determine the angle by which the canoe's heading changed.

(f) Finally, update the initial heading of 310 degrees with the angle of change to find the new heading of the canoe after the nap.

Answer. 250°

92. After spending 6 weeks in Samoa, vaka Paikea is making his way back from Apia to Rarotonga. On July 14, 2023, the canoe is sailing with a heading of 50°, and the wind is coming to the canoe from 60° to the right of the your heading. What is the heading and house from which the wind is coming?

Answer. 110°; ʻĀina Malanai

1.3 Unit Circle

In this section, we will introduce the trigonometric functions using the Unit Circle.

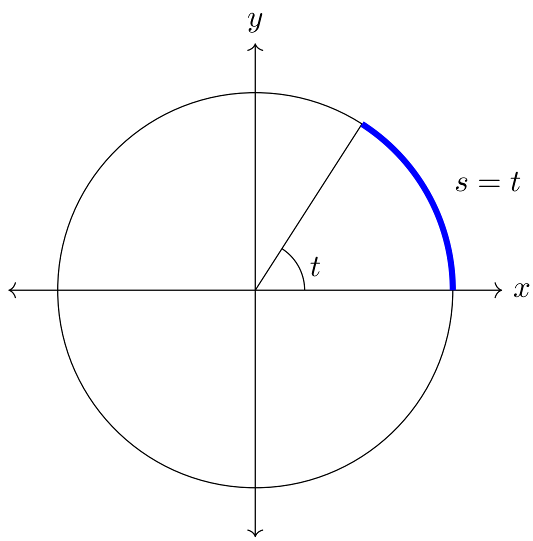
1.3.1 Unit Circle

Definition 1.3.1 Unit Circe. The **unit circle** is a circle whose radius is 1 and whose center is at the origin of a rectangular plane (or xy-plane). The equation for the unit circle is

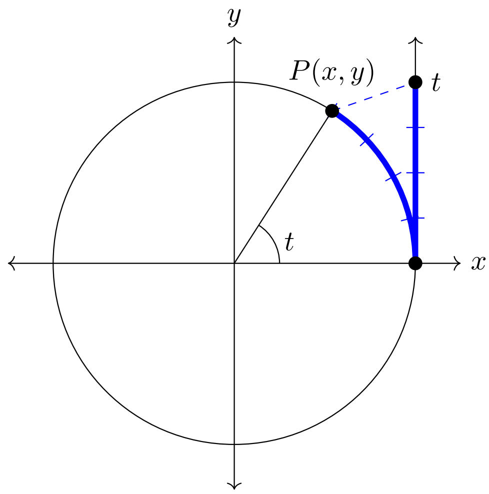
$$x^2 + y^2 = 1$$

$\Diamond$

Let t be a real number. Recall from Definition 1.2.18 that a radian measure of a central angle, t, is defined as the ratio of the arc length s to the radius r. In other words, $t = \frac{s}{r}$. In the unit circle, the radius is $r = 1$, and the angle in radians is equal to the arc length, $t = s$. We will let t be in radians. The circumference of the unit circle is $2\pi r = 2\pi \cdot 1 = 2\pi$.



If $t \geq 0$, we can imagine wrapping a line around the unit circle, marking off a distance of t in a counterclockwise direction, and labeling that point $P(x, y)$, whic becomes the terminal point. If $t < 0$ then we would wrap in a clockwise direction.



If $t > 2\pi$ or $t < -2\pi$, then the length is longer than the circumference of the unit circle and you will need to travel around the unit circle more than once before arrive at the point $P(x, y)$. Therefore, we can conclude that regardless of the value of t, we have a unique point $P(x, y)$ that lies on the unit circle. We call $P(x, y)$ the *point on the unit circle that corresponds to* t.

1.3.2 Trigonometric Functions

The x- and y-coordinates for $P(x, y)$ can then be used to define the six trigonometric functions of a real number t:

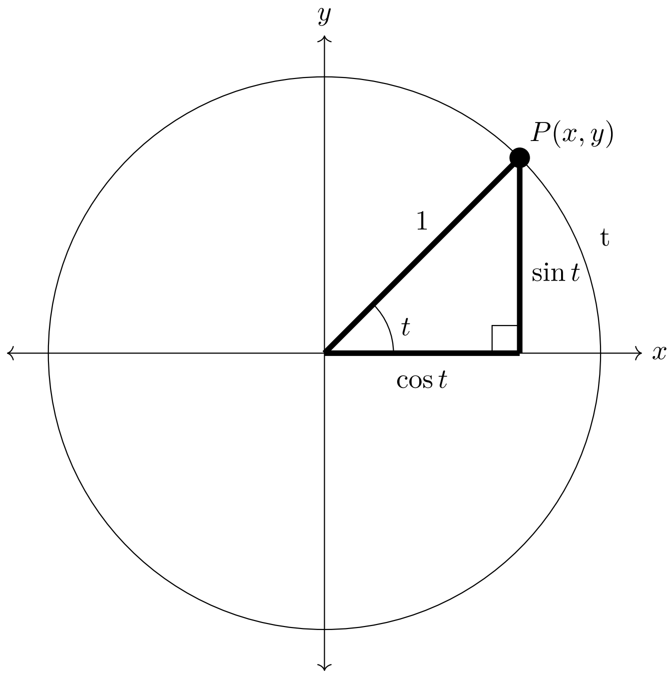
sine *cosine* *tangent* *cosecant* *secant* *cotangent*

which are abbreviated as *sin*, *cos*, *tan*, *csc*, *sec*, and *cot*, respectively.

Definition 1.3.2 Definition of Trigonometric Functions. Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle associated with t. Then

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x},\ (x \neq 0)$$

$$\csc t = \frac{1}{y},\ (y \neq 0) \qquad \sec t = \frac{1}{x},\ (x \neq 0) \qquad \cot t = \frac{x}{y},\ (y \neq 0)$$



Notice that $\tan t$ and $\sec t$ re undefined when $x = 0$ and $\csc t$ and $\cot t$ are undefined when $y = 0$. $\Diamond$

Example 1.3.3 Let t be the angle that corresponds to the point $P(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Find the exact values of the six trigonometric functions corresponding to t: $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, $\cot t$.

Solution. The point $P(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ gives us $x = \frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$. Then we have

$$
\begin{aligned}
\sin\theta &= y = -\frac{1}{2}, & \csc\theta &= \frac{1}{y} = \frac{1}{-\frac{1}{2}} = -2,\\
\cos\theta &= x = \frac{\sqrt{3}}{2}, & \sec\theta &= \frac{1}{x} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3},\\
\tan\theta &= \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}, & \cot\theta &= \frac{x}{y} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}.
\end{aligned}
$$

$\square$

1.3.3 Trigonometric Functions of an Angle

Definition 1.3.4 Trigonometric Functions of an Angle. If θ is an angle with radian measure t, then the **six trigonometric functions** become

$$
\begin{aligned}
&\sin\theta = y & &\cos\theta = x & &\tan\theta = \frac{y}{x},\ (x \neq 0)\\
&\csc\theta = \frac{1}{y},\ (y \neq 0) & &\sec\theta = \frac{1}{x},\ (x \neq 0) & &\cot\theta = \frac{x}{y},\ (y \neq 0)
\end{aligned}
$$

$\Diamond$

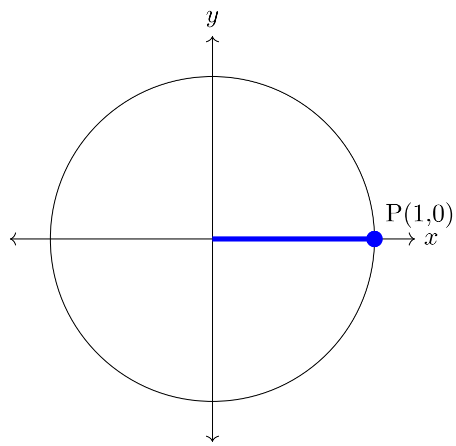
Example 1.3.5 Find the exact values of the six trigonometric functions for

1. $\theta = 0$

2. $\theta = \dfrac{3\pi}{2}$
3. $\theta = 5\pi$

Solution.

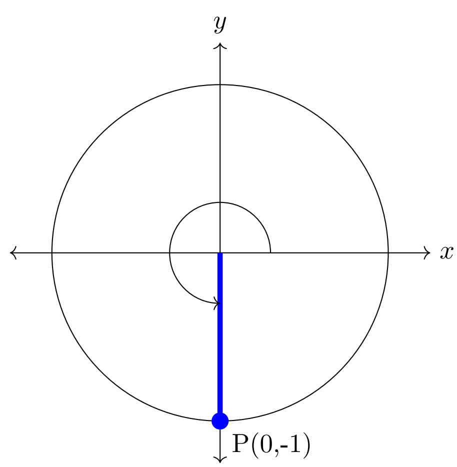
1. When $\theta = 0$ radians $(0°)$, the point on the circle is $P(1, 0)$.



Then $x = 1$ and $y = 0$ gives us

$$\begin{aligned} \sin 0 &= \sin 0° = 0, & \csc 0 &= \csc 0° = \text{undefined}, \\ \cos 0 &= \cos 0° = 1, & \sec 0 &= \sec 0° = 1, \\ \tan 0 &= \tan 0° = 0, & \cot 0 &= \cot 0° = \text{undefined}. \end{aligned}$$

2. When $\theta = \frac{3\pi}{2}$ radians $(270°)$, the point on the circle is $P(0, -1)$.



Then $x = 0$ and $y = -1$ gives us

$$\begin{aligned} \sin \frac{3\pi}{2} &= \sin 270° = -1, & \csc \frac{3\pi}{2} &= \csc 270° = -1, \\ \cos \frac{3\pi}{2} &= \cos 270° = 0, & \sec \frac{3\pi}{2} &= \sec 270° = \text{undefined}, \\ \tan \frac{3\pi}{2} &= \tan 270° = \text{undefined}, & \cot \frac{3\pi}{2} &= \cot 270° = 0 \end{aligned}$$

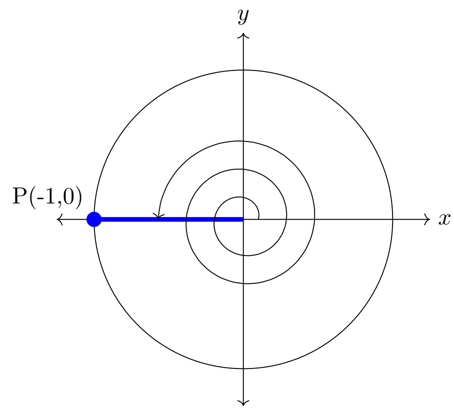
3. Since $\theta = 5\pi > 2\pi$, our angle is greater than one full rotation of a circle. We first subtract θ by one rotation, 2π, to get

$$5\pi - 2\pi - = 3\pi$$

Once again, since we have completed more than one full rotation, we can repeat the previous step:

$$3\pi - 2\pi = \pi$$

The values of the six trigonometric functions when $\theta = 5\pi$ are equal to those when $\theta = \pi$. Notice that 5π and π are *coterminal angles*, both ending at the point$P(-1,0)$.



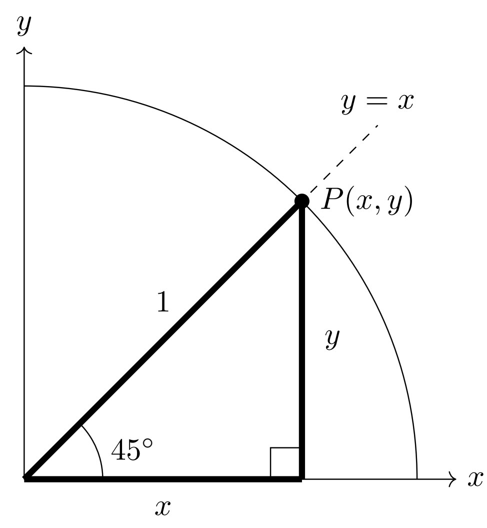
Since $x = -1$ and $y = 0$ we have

$$\begin{aligned} &\sin 5\pi = 0, && \cos 5\pi = -1, && \tan 5\pi = 0, \\ &\csc 5\pi = \text{undefined}, && \sec 5\pi = -1, && \cot 5\pi = \text{undefined}. \end{aligned}$$

□

Example 1.3.6 Finding the Exact Values of the Trigonometric Functions for $\theta = 45°$. Find the exact values of the six trigonometric functions for $\theta = 45°$.

Solution. We begin by drawing a right triangle with a base angle of 45° in the unit circle.



Since the first quadrant has 90°, at $\theta = 45°$, the point P lies on the line that bisects the first quadrant. This means the point P is on the line $y = x$. Since $P(x, y)$ also lies on the unit circle, whose equation is $x^2 + y^2 = 1$, we get

$$x^2 + y^2 = 1$$

$$
\begin{aligned}
x^2 + x^2 &= 1 && \text{(since } y = x\text{)} \\
2x^2 &= 1 \\
x^2 &= \frac{1}{2} \\
x &= \frac{1}{\sqrt{2}} \\
y &= \frac{1}{\sqrt{2}} && \text{(since } y = x\text{)}
\end{aligned}
$$

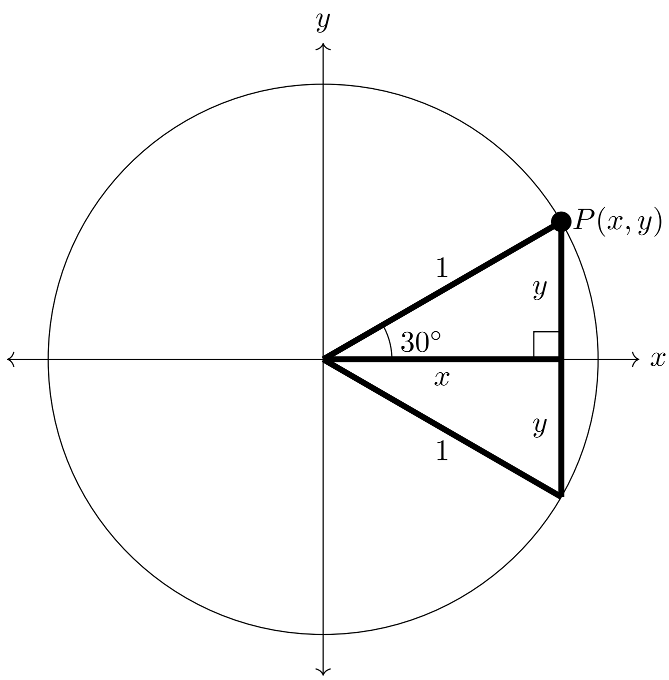
Then

$$
\begin{aligned}
\sin 45^\circ &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} & \csc 45^\circ &= \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \\
\cos 45^\circ &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} & \sec 45^\circ &= \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \\
\tan 45^\circ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 & \cot 45^\circ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1
\end{aligned}
$$

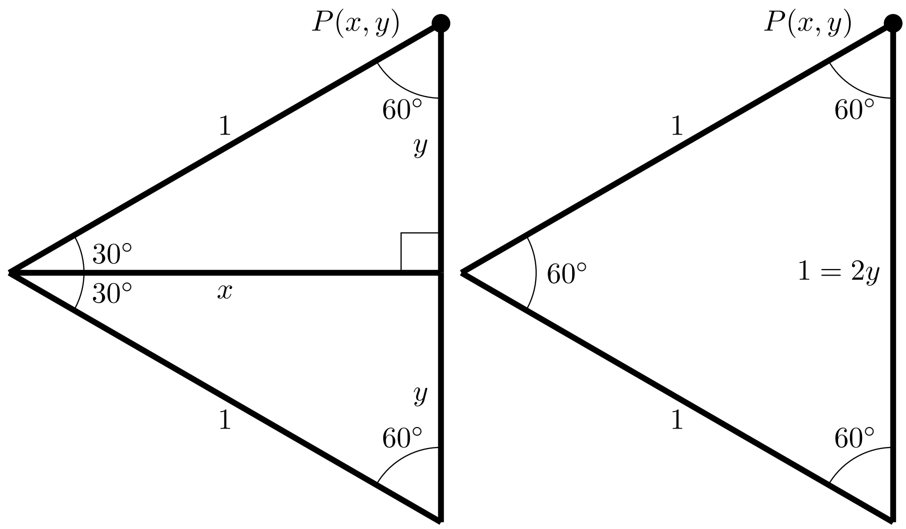
□

Example 1.3.7 Finding the Exact Values of the Trigonometric Functions for $\theta = 30^\circ$. Find the exact values of the six trigonometric functions for $\theta = 30^\circ$.

Solution. First, we will draw a triangle in a circle with an angle of 30° and a second triangle with an angle of -30° .



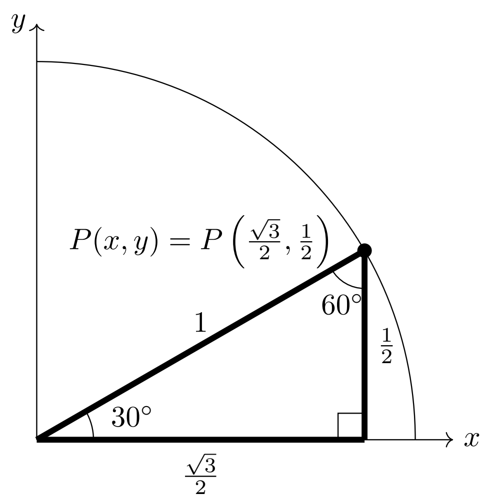
This gives us two 30-60-90 triangles. Notice this now gives us one larger triangle whose angles are all 60°. Thus we have an equilateral triangle, with each side of length 1.



We see that $1 = 2y$ so $y = \frac{1}{2}$. Then by the Pythagorean Theorem,

$$\begin{aligned} x^2 + y^2 &= 1^2 \\ x^2 + \left(\frac{1}{2}\right)^2 &= 1 \\ x^2 + \frac{1}{4} &= 1 \\ x^2 &= \frac{3}{4} \\ x &= \frac{\sqrt{3}}{2} \end{aligned}$$

Giving us the following triangle



Then

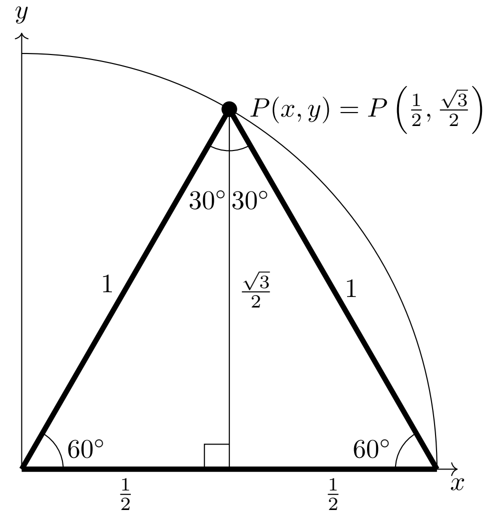
$$\sin 30^\circ = \frac{1}{2}, \qquad \csc 30^\circ = \frac{1}{\frac{1}{2}} = 2,$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \sec 30^\circ = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

$$\tan 30° = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \qquad \cot 30° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

□

Remark 1.3.8 Finding the Exact Values of the Trigonometric Functions for $\theta = 90°$. Similarly, we can get the following for $\theta = 60°$.



We now summarize what we know about the six trigonometric functions for special angles. Note the trigonometric functions for $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{3}$ are left as exercises.

Table 1.3.9 Trigonometric functions for special angles

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	0	1	0	undef	1	undef
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	undef	1	undef	0

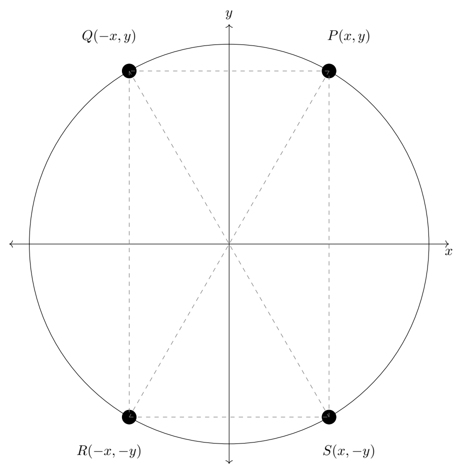
1.3.4 Symmetry on the Unit Circle

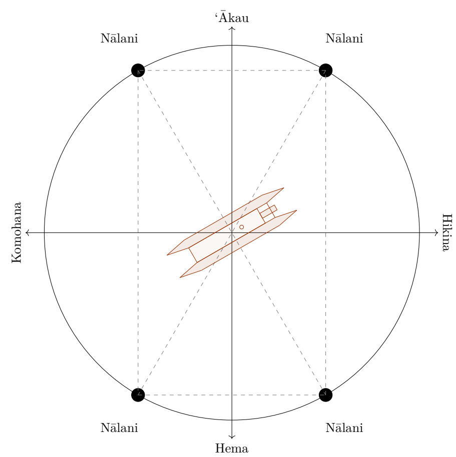
If the point $P(x, y)$ lies on the unit circle, the following symmetric points also lie on the unit circle:

1. $Q(-x, y)$: Symmetry about the y-axis.
2. $R(-x, -y)$: Symmetry about the origin.

3. $S(x, -y)$: Symmetry about the x-axis.

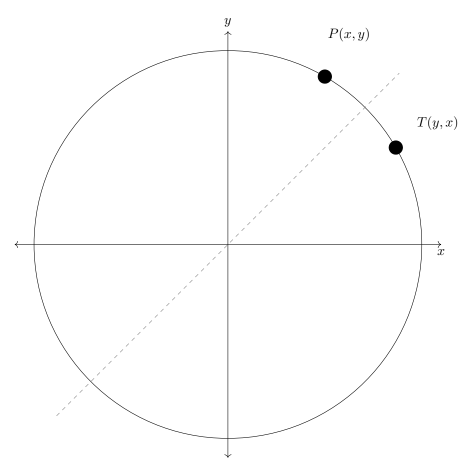
This symmetry within the unit circle resembles the pattern observed in the Star Compass. When a star emerges in the eastern sky, it will eventually descend and set in the corresponding house of the western sky. For instance, if a star rises above the horizon in the Nālani house of the Koʻolau quadrant (northeast), it will journey across the sky and set in the equivalent house within the Hoʻolua quadrant (northwest). This similarity aligns with the symmetry between points $P(x, y)$ and $Q(-x, y)$. Additionally, if an ocean swell or wind originates from the Nālani house in the Malanai quadrant (southeast), it will pass the waʻa and exit in the opposite direction toward the Hoʻolua quadrant (northwest), still within the Nālani house. This mirrors the symmetry between points $S(x, -y)$ and $Q(-x, y)$.



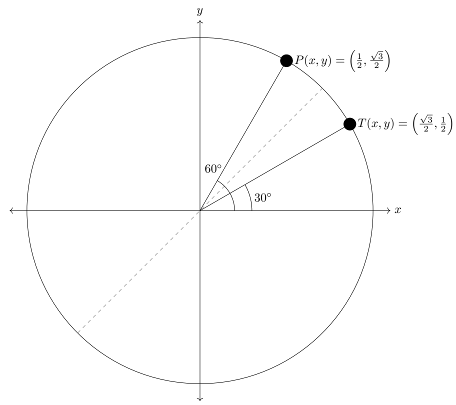


A fourth form of symmetry involves reflecting points across the diagonal line $y = x$, where the x- and y-values are equal.

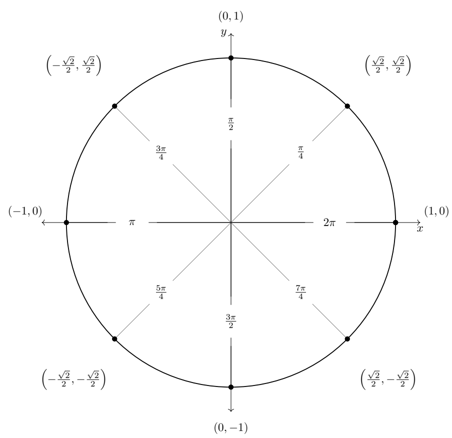
1. $T(y, x)$: Symmetry about the line $y = x$. This is accomplished by interchanging the x- and y-values.

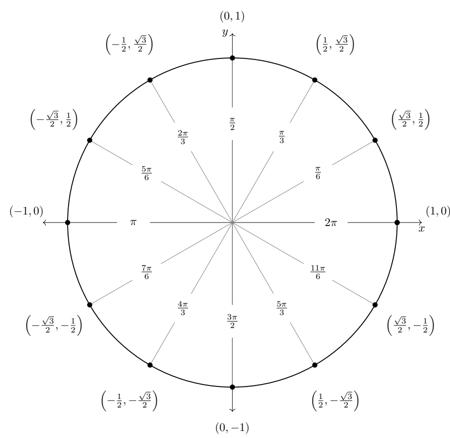


Notice on the Unit Circle that the radius extending from the center at an angle of 30° to the point $T(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is symmetric about the line $y = x$, in relation to the radius extending from the center at an angle of 60° to the point $P(x, y)$.



Using symmetry about the x-axis, symmetry about the y-axis, and symmetry about the origin, we can complete the unit circle, as long as we remember that the x-values in Quadrants II and III are negative while the y-values in Quadrants III and IV are negative.





Finally, we tie everything together and look at the entire Unit Circle. At first glance it may seem intimidating, however, similar to the Star Compass, there is a lot of symmetry (x-axis, y-axis, origin, about the line $y = x$) and it can help by focusing on one quadrant, and use symmetry to fill out the rest of the circle.

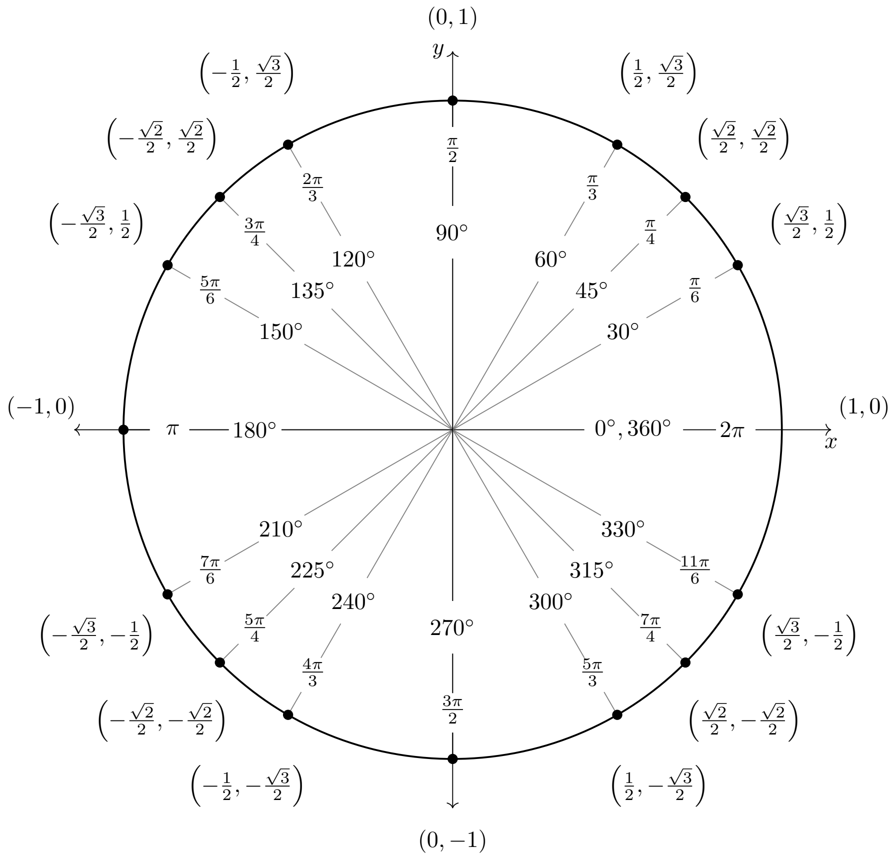


Figure 1.3.10 The Unit Circle for common angles in radians and degrees.

1.3.5 Trigonometric Functions on a Circle with Radius r

Until now, computing the exact values of trigonometric functions of an angle θ required us to locate the corresponding point $P(x, y)$ on the unit circle. However, we can use any circle with center at the origin, that is, any circle of the form $x^2 + y^2 = r^2$, where $r > 0$ is the radius. Note that if $r = 1$, then it is the unit circle.

Theorem 1.3.11 *For an angle θ in standard position, let $P(x, y)$ be the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. Then*

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ (x \neq 0)$$
$$\csc\theta = \frac{r}{y},\ (y \neq 0) \qquad \sec\theta = \frac{r}{x},\ (x \neq 0) \qquad \cot\theta = \frac{x}{y},\ (y \neq 0)$$

1.3.6 Exercises

Exercise Group. Verify algebraically that the point P is on the unit circle $(x^2 + y^2 = 1)$

1. $P\left(\frac{3}{5}, -\frac{4}{5}\right)$

Answer. $\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = 1$

2. $P\left(-\frac{\sqrt{39}}{8}, -\frac{5}{8}\right)$

Answer. $\left(-\frac{\sqrt{39}}{8}\right)^2 + \left(-\frac{5}{8}\right)^2 = 1$

3. $P\left(-\frac{\sqrt{55}}{8}, \frac{3}{8}\right)$

Answer. $\left(-\frac{\sqrt{55}}{8}\right)^2 + \left(\frac{3}{8}\right)^2 = 1$

4. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

Answer. $\left(-\frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2 = 1$

5. $P\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

Answer. $\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2 = 1$

6. $P\left(\frac{\sqrt{21}}{5}, -\frac{2}{5}\right)$

Answer. $\left(\frac{\sqrt{21}}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 = 1$

Exercise Group. Let the point P be on the unit circle. Given the quadrant that P lies in, determine the missing coordinate, a

7. III; $P\left(-\frac{2}{3}, a\right)$

Answer. $-\frac{\sqrt{5}}{3}$

8. IV; $P\left(\frac{5}{8}, a\right)$

Answer. $-\frac{\sqrt{39}}{8}$

9. III; $P\left(a, -\frac{2}{5}\right)$

Answer. $-\frac{\sqrt{21}}{5}$

10. II; $P\left(a, \frac{4}{9}\right)$

Answer. $-\frac{\sqrt{65}}{9}$

Exercise Group. Given an angle θ that corresponds to the point P on the unit circle, determine the coordinates of the point $P(x, y)$.

11. $\theta = \frac{\pi}{2}$

Answer. $(0, 1)$

12. $\theta = \pi$

Answer. $(-1, 0)$

13. $\theta = \frac{5\pi}{3}$

Answer. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

14. $\theta = \frac{4\pi}{3}$

Answer. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

15. $\theta = -\frac{\pi}{4}$

Answer. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

16. $\theta = \frac{5\pi}{6}$

Answer. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

17. $\theta = 315°$

Answer. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

18. $\theta = 720°$

Answer. $(1, 0)$

19. $\theta = 60°$

Answer. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

20. $\theta = -180°$

Answer. $(-1, 0)$

21. $\theta = 210°$

Answer. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

22. $\theta = 120°$

Answer. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Exercise Group. For each angle θ in Exercises 1.3.6.11–22, find the exact values of the six trigonometric functions. If any are not defined, say "*undefined*."

23. $\theta = \frac{\pi}{2}$

Answer. $\sin\frac{\pi}{2} = 1$; $\cos\frac{\pi}{2} = 0$; $\tan\frac{\pi}{2}$ is undefined; $\csc\frac{\pi}{2} = 1$; $\sec\frac{\pi}{2}$ is undefined; $\cot\frac{\pi}{2} = 0$

24. $\theta = \pi$

Answer. $\sin\pi = 0$; $\cos\pi = -1$; $\tan\pi = 0$; $\csc\pi$ is undefined; $\sec\pi = -1$; $\cot\pi$ is undefined

25. $\theta = \frac{5\pi}{3}$

Answer. $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$; $\cos\frac{5\pi}{3} = \frac{1}{2}$; $\tan\frac{5\pi}{3} = -\sqrt{3}$; $\csc\frac{5\pi}{3} = -\frac{2\sqrt{3}}{3}$; $\sec\frac{5\pi}{3} = 2$; $\cot\frac{5\pi}{3} = -\frac{\sqrt{3}}{3}$

26. $\theta = \frac{4\pi}{3}$

Answer. $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$; $\cos\frac{4\pi}{3} = -\frac{1}{2}$; $\tan\frac{4\pi}{3} = \sqrt{3}$; $\csc\frac{4\pi}{3} = -\frac{2\sqrt{3}}{3}$; $\sec\frac{4\pi}{3} = -2$; $\cot\frac{4\pi}{3} = \frac{\sqrt{3}}{3}$

27. $\theta = -\frac{\pi}{4}$

Answer. $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$; $\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $\tan\left(-\frac{\pi}{4}\right) = -1$; $\csc\left(-\frac{\pi}{4}\right) = -\sqrt{2}$; $\sec\left(-\frac{\pi}{4}\right) = \sqrt{2}$; $\cot\left(-\frac{\pi}{4}\right) = -1$

28. $\theta = \frac{5\pi}{6}$

Answer. $\sin\frac{5\pi}{6} = \frac{1}{2}$; $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$; $\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$; $\csc\frac{5\pi}{6} = 2$; $\sec\frac{5\pi}{6} = -\frac{2\sqrt{3}}{3}$; $\cot\frac{5\pi}{6} = -\sqrt{3}$

29. $\theta = 315°$

Answer. $\sin 315° = -\frac{\sqrt{2}}{2}$; $\cos 315° = \frac{\sqrt{2}}{2}$; $\tan 315° = -1$; $\csc 315° = -\sqrt{2}$; $\sec 315° = \sqrt{2}$; $\cot 315° = -1$

30. $\theta = 720°$

Answer. $\sin 720° = 0$; $\cos 720° = 1$; $\tan 720° = 0$; $\csc 720°$ is undefined; $\sec 720° = 1$; $\cot 720°$ is undefined

31. $\theta = 60°$

Answer. $\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \sqrt{3}$; $\csc 60° = \frac{2\sqrt{3}}{3}$; $\sec 60° = 2$; $\cot 60° = \frac{\sqrt{3}}{3}$

32. $\theta = -180°$

Answer. $\sin(-180°) = 0$; $\cos(-180°) = -1$; $\tan(-180°) = 0$; $\csc(-180°)$ is undefined; $\sec(-180°) = -1$; $\cot(-180°)$ is undefined

33. $\theta = 210°$

Answer. $\sin 210° = -\frac{1}{2}$; $\cos 210° = -\frac{\sqrt{3}}{2}$; $\tan 210° = \frac{\sqrt{3}}{3}$; $\csc 210° = -2$; $\sec 210° = -\frac{2\sqrt{3}}{3}$; $\cot 210° = \sqrt{3}$

34. $\theta = 120°$

Answer. $\sin 120° = \frac{\sqrt{3}}{2}$; $\cos 120° = -\frac{1}{2}$; $\tan 120° = -\sqrt{3}$; $\csc 120° = \frac{2\sqrt{3}}{3}$; $\sec 120° = -2$; $\cot 120° = -\frac{\sqrt{3}}{3}$

Exercise Group. Let θ be the angle that corresponds to the point P. Exercises 1.3.6.1–6 verified P is on the unit circle. Find the exact values of the six trigonometric functions of θ.

35. $P\left(\frac{3}{5}, -\frac{4}{5}\right)$

Answer. $\sin\theta = -\frac{4}{5}$; $\cos\theta = \frac{3}{5}$; $\tan\theta = -\frac{4}{3}$; $\csc\theta = -\frac{5}{4}$; $\sec\theta = \frac{5}{3}$; $\cot\theta = -\frac{3}{4}$

36. $P\left(-\frac{\sqrt{39}}{8}, -\frac{5}{8}\right)$

Answer. $\sin\theta = -\frac{5}{8}$; $\cos\theta = -\frac{\sqrt{39}}{8}$; $\tan\theta = \frac{5\sqrt{39}}{39}$; $\csc\theta = -\frac{8}{5}$; $\sec\theta = -\frac{8\sqrt{39}}{39}$; $\cot\theta = \frac{\sqrt{39}}{5}$

37. $P\left(-\frac{\sqrt{55}}{8}, \frac{3}{8}\right)$

Answer. $\sin\theta = \frac{3}{8}$; $\cos\theta = -\frac{\sqrt{55}}{8}$; $\tan\theta = -\frac{3\sqrt{55}}{55}$; $\csc\theta = \frac{8}{3}$; $\sec\theta = -\frac{8\sqrt{55}}{55}$; $\cot\theta = -\frac{\sqrt{55}}{3}$

38. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

Answer. $\sin\theta = \frac{\sqrt{5}}{3}$; $\cos\theta = -\frac{2}{3}$; $\tan\theta = -\frac{\sqrt{5}}{2}$; $\csc\theta = \frac{3\sqrt{5}}{5}$; $\sec\theta = -\frac{3}{2}$; $\cot\theta = -\frac{2\sqrt{5}}{5}$

39. $P\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

Answer. $\sin\theta = \frac{\sqrt{7}}{4}$; $\cos\theta = \frac{3}{4}$; $\tan\theta = \frac{\sqrt{7}}{3}$; $\csc\theta = \frac{4\sqrt{7}}{7}$; $\sec\theta = \frac{4}{3}$; $\cot\theta = \frac{3\sqrt{7}}{7}$

40. $P\left(\frac{\sqrt{21}}{5}, -\frac{2}{5}\right)$

Answer. $\sin\theta = -\frac{2}{5}$; $\cos\theta = \frac{\sqrt{21}}{5}$; $\tan\theta = -\frac{2\sqrt{21}}{21}$; $\csc\theta = -\frac{5}{2}$; $\sec\theta = \frac{5\sqrt{21}}{21}$; $\cot\theta = -\frac{\sqrt{21}}{2}$

Exercise Group. Find the exact value of each expression.

41. $\sin 30° + \sin 150°$

Answer. 1

42. $\cos 30° + \cos 150°$

Answer. 0

43. $\sin 60° + \sin 120° + \sin 240° + \sin 300°$

Answer. 0

44. $\cos 60° + \cos 120° + \cos 240° + \cos 300°$

Answer. 0

45. $\tan 45° + \tan 135°$

Answer. 0

46. $\tan 135° + \tan 225°$

Answer. 0

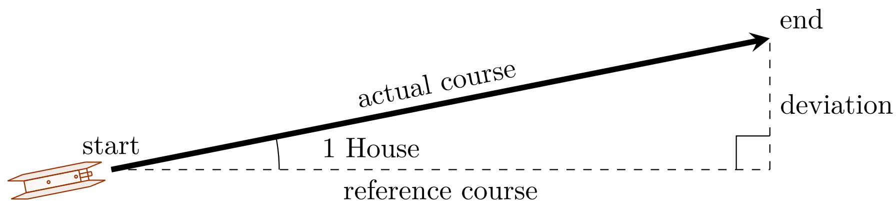
47. $\tan 225° + \tan 315°$

Answer. 0

48. $\tan 45° + \tan 225°$

Answer. 2

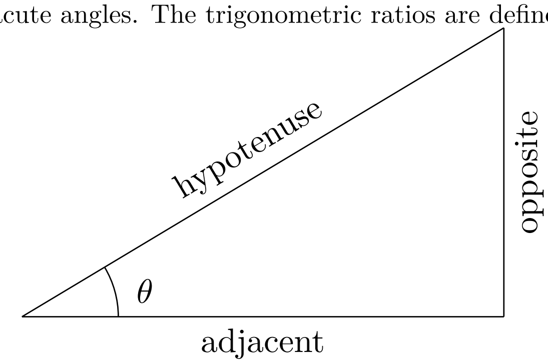
1.4 Right Triangle Trigonometry



During a voyage, a navigator utilizes a *reference course* —a line connecting the starting point and destination—to monitor their position. When the wa‘a (canoe) encounters winds that veer it off course, the navigator mentally plots their position relative to the reference course. To ensure the destination isn't missed, navigators must monitor their deviation from the intended course, involving measurement of the angle of deviation from the reference course (in units of houses) and determining the distance traveled. This section explores the calculation of trigonometric functions using right triangles, enabling us to assess how much the wa‘a has strayed from its intended reference course.

1.4.1 Trigonometric Ratios

Definition 1.4.1 Trigonometric Ratios. Consider a right triangle with θ as one of its acute angles. The trigonometric ratios are defined as follows:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

A common mnemonic for remembering these relationships is SOHCAHTOA, formed from the first letters of "Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, Tangent is Opposite over Adjacent." ◊

Based on the definition of the six trigonometric functions, we have the following trigonometric identities.

Definition 1.4.2 Reciprocal Identities.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

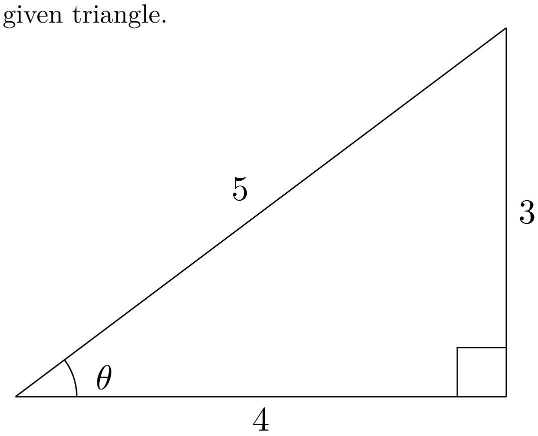
◊

Definition 1.4.3 Quotient Identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

◊

Example 1.4.4 Find the exact values of the six trigonometric ratios of the angle θ in the given triangle.

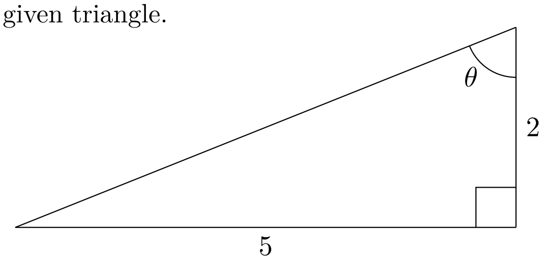


Solution. By the definition of the trigonometric ratios, we have

$$\sin\theta = \frac{3}{5} \qquad \cos\theta = \frac{4}{5} \qquad \tan\theta = \frac{3}{4}$$
$$\csc\theta = \frac{5}{3} \qquad \sec\theta = \frac{5}{4} \qquad \cot\theta = \frac{4}{3}$$

□

Example 1.4.5 Find the exact values of the six trigonometric ratios of the angle θ in the given triangle.



Solution. Notice that θ is in a different position. Here, the adjacent side is 3 and the opposite side is 5. If we let h denote the hypotenuse, then we can use the Pythagorean Theorem to get

$$h = \sqrt{5^2 + 2^2} = \sqrt{29}$$

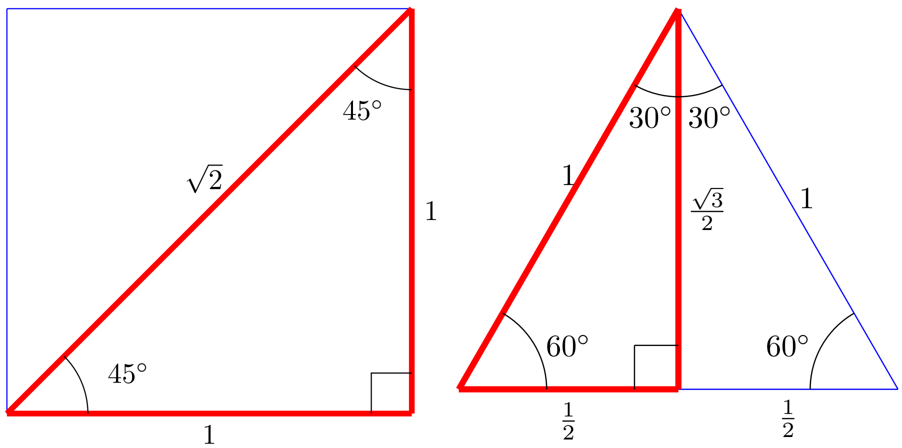
Then by the definition of the trigonometric ratios, we have

$$\sin\theta = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29} \qquad \cos\theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29} \qquad \tan\theta = \frac{5}{2}$$
$$\csc\theta = \frac{\sqrt{29}}{5} \qquad \sec\theta = \frac{\sqrt{29}}{2} \qquad \cot\theta = \frac{2}{5}$$

□

1.4.2 Special Triangles

The angles $30°, 45°, 60°$ $(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3})$ give special values for trigonometric functions. The following figures are used to calculate trigonometric values.



The trigonometric values for the special angles $0, 30°, 45°, 60°, 90°$ $\left(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ are given in Table 1.4.6.

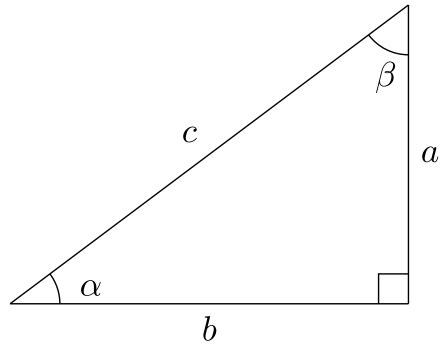
Table 1.4.6 Values of the trigonometric functions in Quadrant I

θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined

1.4.3 Cofunctions

The symmetry between $\sin\theta$ and $\cos\theta$ becomes evident when reversing the order of sine and cosine values from $90°$ to $0°$. This symmetry yields $\sin 0° = \cos 90°$, $\sin 30° = \cos 60°$, $\sin 45° = \cos 45°$, $\sin 60° = \cos 30°$, and $\sin 90° = \cos 0°$.

This pattern between sine and cosine is no coincidence; it emerges because the three angles in a triangle add up to $180°$ or π radians. When considering a right triangle, the remaining two angles combine to form $90°$ or $\frac{\pi}{2}$ radians, making them *complementary angles*.



Consider the right triangle in the figure above, where angles α and β are complementary angles. Side $a$is opposite of angle α, and side b is opposite of angle β. Notice that we can also describe side b as adjacent to angle α and side a as adjacent to angle β. Therefore,

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \text{ and } \cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

Thus we can conclude that

$$\sin\alpha = \frac{a}{c} = \cos\beta$$

Sine and cosine are called **cofunctions** because of this relationship between these functions and their complementary angles. We can obtain similar relationships for all trigonometric functions:

$$\begin{array}{lll} \sin\alpha = \frac{a}{c} = \cos\beta & \cos\alpha = \frac{b}{c} = \sin\beta & \tan\alpha = \frac{a}{b} = \cot\beta \\ \csc\alpha = \frac{c}{a} = \sec\beta & \sec\alpha = \frac{c}{b} = \csc\beta & \cot\alpha = \frac{b}{a} = \tan\beta \end{array}$$

Since α and β are complementary angles, $\alpha + \beta = 90°$. Rearranging, we get $\beta = 90° - \alpha$. Substituting this into our cofunctions and replacing α with θ, we get our cofunction identities.

Definition 1.4.7 Cofunction Identities. The cofunction identities in degrees are

$$\begin{array}{lll} \sin\theta = \cos(90° - \theta) & \cos\theta = \sin(90° - \theta) & \tan\theta = \cot(90° - \theta) \\ \csc\theta = \sec(90° - \theta) & \sec\theta = \csc(90° - \theta) & \cot\theta = \tan(90° - \theta) \end{array}$$

The cofunction identities in radians are

$$\begin{array}{lll} \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) & \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) & \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \\ \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) & \sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) & \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right) \end{array}$$

◊

Example 1.4.8 The Cofunction Identities explains the symmetry in Table 1.4.6
.

$$\begin{array}{rr} \sin 0° = \cos(90° - 0°) = \cos 90° & \sin 60° = \cos(90° - 60°) = \cos 30° \\ \sin 30° = \cos(90° - 30°) = \cos 60° & \sin 90° = \cos(90° - 90°) = \cos 0° \\ \sin 45° = \cos(90° - 45°) = \cos 45° & \end{array}$$

□

Remark 1.4.9 Patterns in the Trigonometric Table. Learning the values of the trigonometric functions in this table can increase your confidence and efficiency in trigonometry. To help remember the values of sine and cosine, we utilize cofunctions and also write them in the form $\sqrt{\cdot}/2$

θ	θ	$\sin\theta$	$\cos\theta$
0	0°	$\sqrt{0}/2$	$\sqrt{4}/2$
$\pi/6$	30°	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	60°	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/2$	90°	$\sqrt{4}/2$	$\sqrt{0}/2$

which simplifies to the values in Table 1.4.6.

1.4.4 Using a Calculator

Sometimes you may encounter an angle other than the special angles described above. In this case, you will have to use a calculator.

First, be sure that your angle is either in degrees or radians, depending on the problem, refer to your calculator's manual for instructions. Most calculators will have a special button for the sine, cosine, and tangent functions. Depending on your calculator, you may see the following keys for

Function	Calculator Key
sine	*SIN*
cosine	*COS*
tangent	*TAN*

To calculate cosecant, secant, and cotangent, you will need to use the identity

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

Answers produced by calculators are *estimates* and you should pay close attention to see if the question is asking for the exact solution or a decimal approximation. For example, if you need to calculate $\sin 45^\circ = \frac{\sqrt{2}}{2}$, the calculator may give the answer as $\sin 45^\circ \approx 0.70710678$, which is a decimal approximation since the actual solution goes on forever. *Unless stated otherwise, answers in the book should be exact*, e.g. $\frac{\sqrt{2}}{2}$ and not 0.70710678.

Example 1.4.10 Use a calculator to evaluate

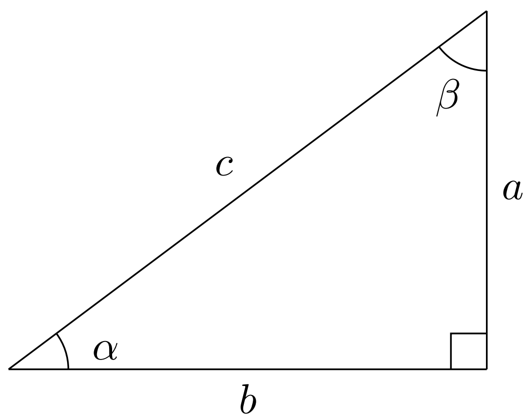
1. $\sin 22^\circ$
2. $\cos 5^\circ$
3. $\cot 53^\circ$
4. $\cos 5$ rad

Solution. Before proceeding, we confirm that our calculator is set to either degree or radian mode. Additionally, for the sake of simplicity, we will round our answers to four decimal places.

1. Input: *SIN* (22); Output: 0.3746
2. Input: *COS* (5); Output: 0.9962
3. Since most calculators do not have a key for cotangent, we Input: (1/ *TAN* (53)); Output: $\frac{1}{1.3270} \approx 0.7536$
4. Since this problem uses radians, we must change the mode on our calculator then Input: *COS* (5); Output: 0.2837.

Observe that $\cos 5° \neq \cos 5$ rad. This emphasizes the significance of verifying whether your calculator is in degree or radian mode. □

1.4.5 Solving Right Triangles



Consider the following right triangle where side a is opposite angle α, side b is opposite angle β, and side c is the hypotenuse. Since α and β are complementary angles, we have

$$\alpha + \beta = 90°$$

Additionally, by the Pythagorean Theorem, we have

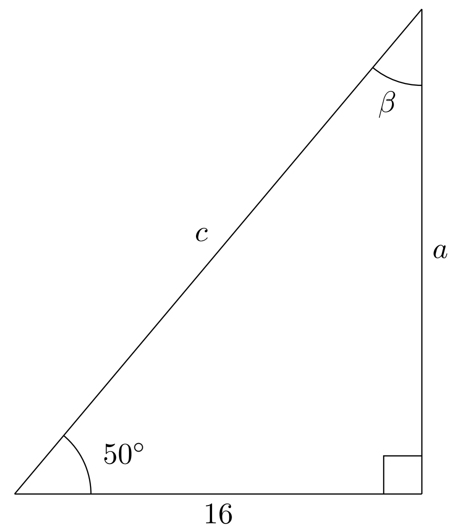
$$a^2 + b^2 = c^2$$

Definition 1.4.11 To **solve a triangle** is the process of determining the values for all three lengths of its sides and the measures of all three angles, based on provided information about the triangle. ♢

Remark 1.4.12 Solving Right Triangles. In solving a right triangle, the following relationships are useful:

$$\alpha + \beta = 90°, \qquad a^2 + b^2 = c^2$$

Example 1.4.13 Solve the right triangle. Round your answer to two decimal places.



Solution. Given that this is a right triangle, we already know one angle is $90°$, and we have an additional angle of$50°$ along with an adjacent side length of 16. To solve this triangle, we need to determine the values of sides a, c, and β. We begin by finding the measure of angle β. Since $50° + \beta = 90°$ we have

$$\beta = 90° - 50° = 40°$$

Next, we will solve for side a. Using the angle $50°$, where the adjacent side is 16 and side a is the side opposite to the angle, we can apply the tangent function, which relates the opposite and adjacent sides:

$$\tan 50° = \frac{a}{16}$$

Multiplying both sides by 16 we get

$$a = 16 \cdot \tan 50° \approx 19.07$$

Using the Pythagorean Theorem, we get

$$c^2 = 16^2 + 19.07^2 \approx 619.66$$

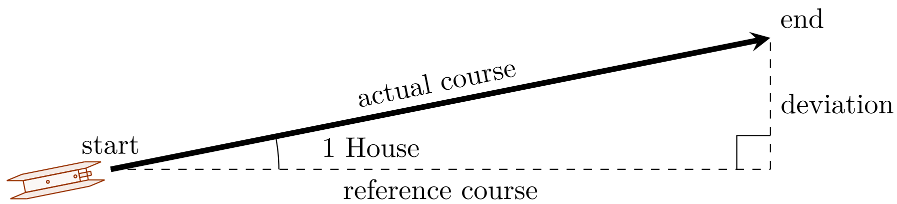
Thus

$$c \approx \sqrt{619.66} \approx 24.89$$

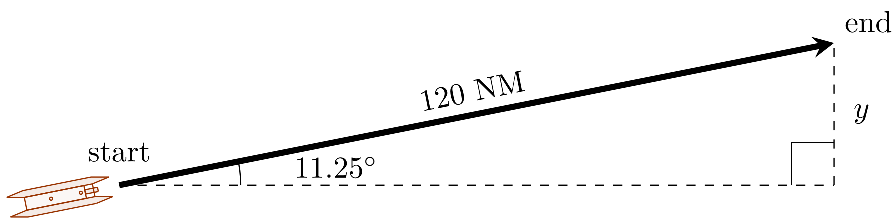
□

1.4.6 Solving Applied Problems

Example 1.4.14 Deviation. We are now ready to calculate the deviation example proposed at the start of this section. In an average day of sailing, a waʻa sails for 120 nautical miles (NM). If Hikianalia is supposed to sail in the direction of Hikina (East), but currents have deviated her course by one house so she actually sailed in the house Lā, how far off the course has Hikianalia deviated?



Solution. From the Star Compass (Figure 1.1.4), the house Lā is one house (11.25°) from Hikina. If we let y denote the distance deviated from the reference course, our right triangle becomes:



Since we know the hypotenuse of the triangle and want to find the side opposite of the angle, we will use the sine function:

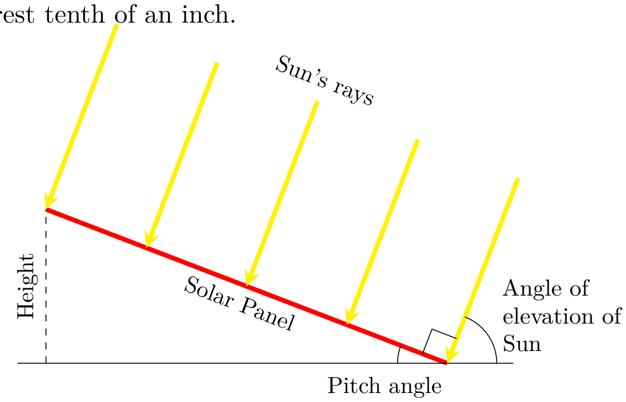
$$\sin 11.25^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{120 \text{ NM}}$$

multiplying both sides by 120, we get

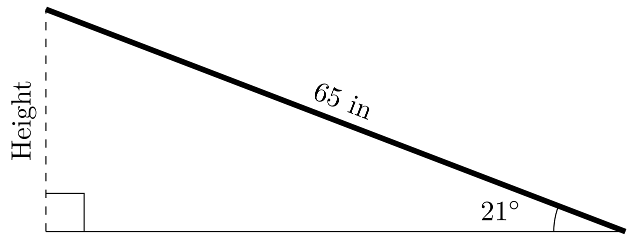
$$y = 120 \cdot \sin 11.25^\circ \text{ NM} \approx 23.4 \text{ NM}$$

So Hikianalia has deviated 23.4 nautical miles north from the reference course. □

Example 1.4.15 Solar panels. Solar panels harness the sun's energy to generate electricity, and for optimal energy output, they should be oriented perpendicularly to the sun's light. The sun's angle of elevation varies based on latitude, and in Hawai‘i, for instance, south-facing solar panels are recommended to have a pitch of 21° to align with the sun's rays. When installing a solar panel, determining its pitch might pose challenges. Instead of measuring the angle directly, an alternative approach involves measuring the height of the panel's top. What height should a south-facing solar panel, measuring 65 inches in length, be installed at to achieve the desired angle of 21°? Round your answer to the nearest tenth of an inch.



Solution. We begin by drawing the triangle.



Since we know the desired angle of pitch of the solar panel and the length of the panel, we can set up the following equation

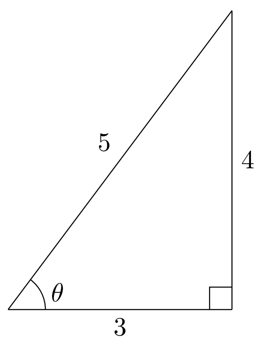
$$\begin{aligned}\sin 21^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{height}}{65 \text{ in}}\\ \text{height} &= 65 \text{ in} \cdot \sin 21^\circ \approx 23.3 \text{ in}\end{aligned}$$

Thus, when installing a solar panel in Hawai'i, the top of the solar panel should be positioned 23.3 inches above the bottom to optimize energy output. □

1.4.7 Exercises

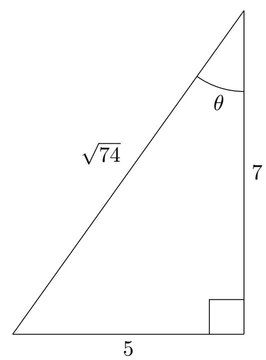
Exercise Group. Find the exact values of the six trigonometric functions of the angle θ in each triangle.

1.



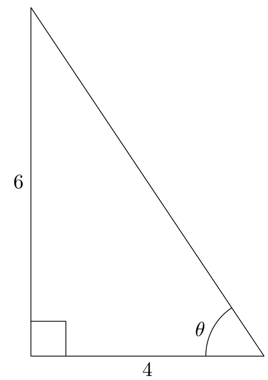
Answer. $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$

2.



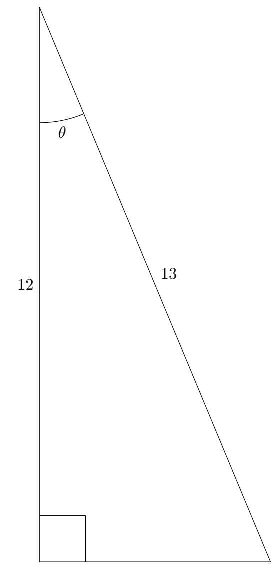
Answer. $\sin\theta = \frac{5\sqrt{74}}{74}$, $\cos\theta = \frac{7\sqrt{74}}{74}$, $\tan\theta = \frac{5}{7}$, $\csc\theta = \frac{\sqrt{74}}{5}$, $\sec\theta = \frac{\sqrt{74}}{7}$, $\cot\theta = \frac{7}{5}$

3.



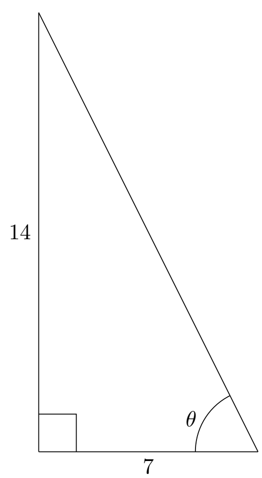
Answer. $\sin\theta = \frac{3\sqrt{13}}{13}$, $\cos\theta = \frac{2\sqrt{13}}{13}$, $\tan\theta = \frac{3}{2}$, $\csc\theta = \frac{\sqrt{13}}{3}$, $\sec\theta = \frac{\sqrt{13}}{2}$, $\cot\theta = \frac{2}{3}$

4.



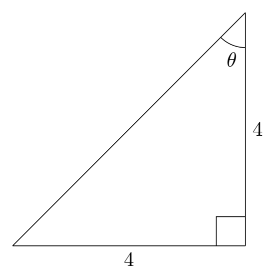
Answer. $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = \frac{12}{5}$

5.



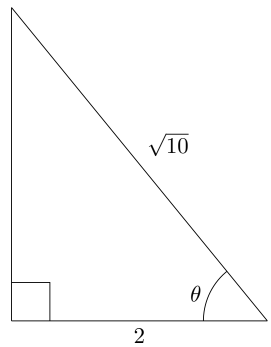
Answer. $\sin\theta = \frac{2\sqrt{5}}{5}$, $\cos\theta = \frac{\sqrt{5}}{5}$, $\tan\theta = 2$, $\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = \sqrt{5}$, $\cot\theta = \frac{1}{2}$

6.



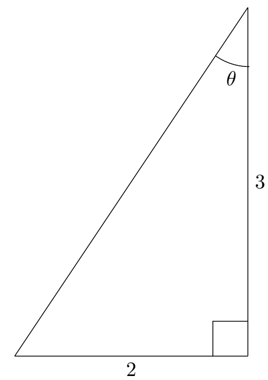
Answer. $\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = 1$, $\csc\theta = \sqrt{2}$, $\sec\theta = \sqrt{2}$, $\cot\theta = 1$

7.



Answer. $\sin\theta = \frac{\sqrt{15}}{5}$, $\cos\theta = \frac{\sqrt{10}}{5}$, $\tan\theta = \frac{\sqrt{6}}{2}$, $\csc\theta = \frac{\sqrt{15}}{3}$, $\sec\theta = \frac{\sqrt{10}}{2}$, $\cot\theta = \frac{\sqrt{6}}{3}$

8.



Answer. $\sin\theta = \frac{2\sqrt{13}}{13}$, $\cos\theta = \frac{3\sqrt{13}}{13}$, $\tan\theta = \frac{2}{3}$, $\csc\theta = \frac{\sqrt{13}}{2}$, $\sec\theta = \frac{\sqrt{13}}{3}$, $\cot\theta = \frac{3}{2}$

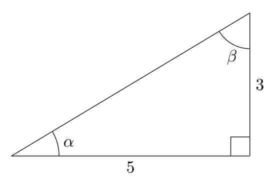
Exercise Group. For each of the following problems, calculate

(a) $\cos\alpha$ and $\sin\beta$;

(b) $\tan\alpha$ and $\cot\beta$;

(c) $\csc\alpha$ and $\sec\beta$

9.



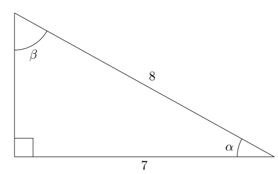
Answer.

(a) $\cos\alpha = \sin\beta = \frac{5\sqrt{34}}{34}$;

(b) $\tan\alpha = \cot\beta = \frac{3}{5}$;

(c) $\csc\alpha = \sec\beta = \dfrac{\sqrt{34}}{3}$

10.



Answer.

(a) $\cos\alpha = \sin\beta = \frac{7}{8}$;

(b) $\tan\alpha = \cot\beta = \frac{\sqrt{15}}{7}$;

(c) $\csc\alpha = \sec\beta = \dfrac{8\sqrt{15}}{15}$

Exercise Group. Use the Cofunction Identities to determine the value of θ

11. $\sin 28° = \cos\theta$

Answer. $\theta = 62°$

12. $\cos 74° = \sin\theta$

Answer. $\theta = 16°$

13. $\tan 52° = \cot\theta$

Answer. $\theta = 62°$

14. $\sec 87° = \csc\theta$

Answer. $\theta = 3°$

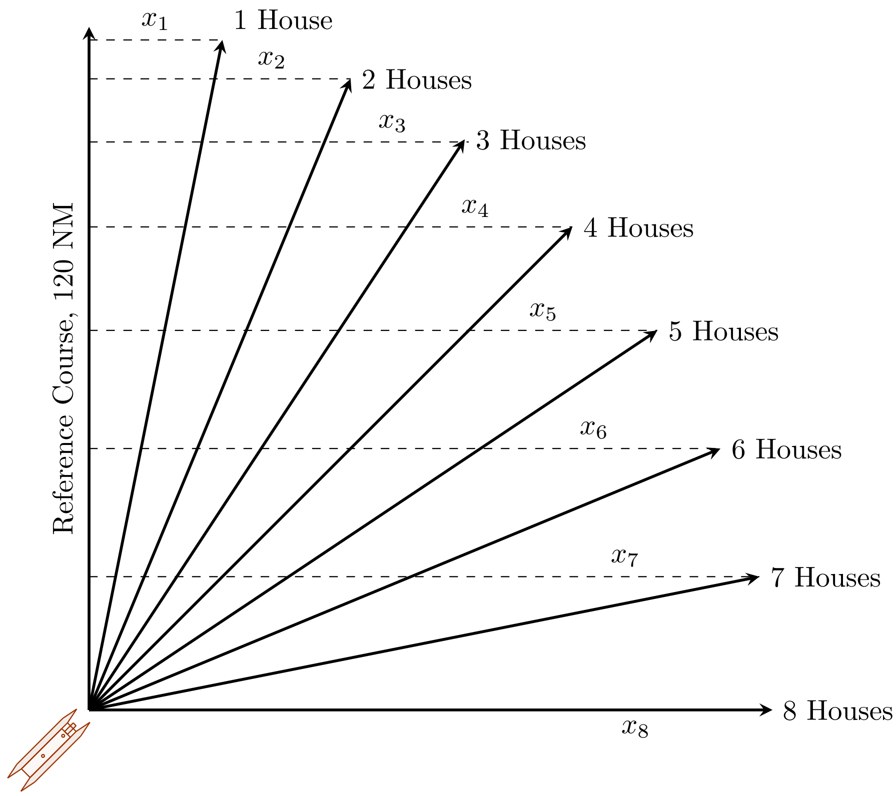
15. $\sin\frac{3\pi}{8} = \cos\theta$

Answer. $\theta = \frac{\pi}{8}$

16. $\cot\frac{2\pi}{5} = \tan\theta$

Answer. $\theta = \frac{\pi}{10}$

Exercise Group. In Example 1.4.14, we determined that when a waʻa sails for one day (120 nautical miles) and deviates from its course by 1 House, the resulting deviation from the reference course is 23.4 NM. Now, calculate the deviations (x) for the remaining 7 angles. Round your answer to the nearest tenth of a nautical mile. Remember that one house corresponds to 11.25°.



17. 2 Houses ($11.25° \times 2 = 22.5°$); x_2

Answer. $x_2 = 45.9$ NM

18. 3 Houses ($11.25° \times 3 = 33.75°$); x_3

Answer. $x_3 = 66.7$ NM

19. 4 Houses ($11.25° \times 4 = 45°$); x_4

Answer. $x_4 = 84.5$ NM

20. 5 Houses ($11.25° \times 5 = 56.25°$); x_5

Answer. $x_5 = 99.8$ NM

21. 6 Houses ($11.25° \times 6 = 67.5°$); x_6

Answer. $x_6 = 110.9$ NM

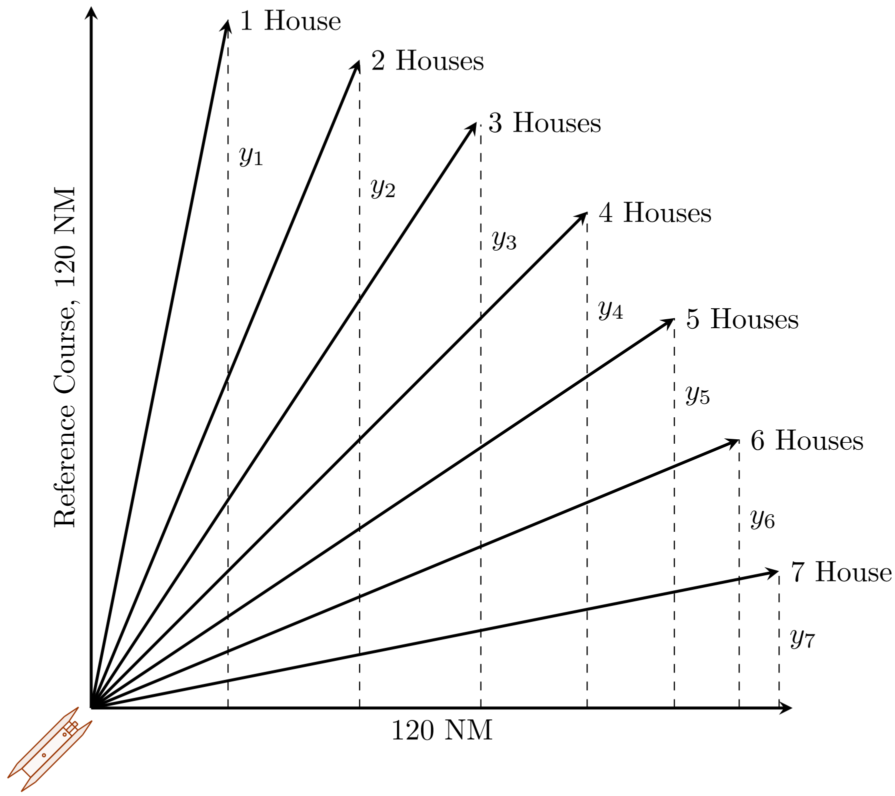
22. 7 Houses ($11.25° \times 7 = 78.75°$); x_7

Answer. $x_7 = 117.7$ NM

23. 8 Houses ($11.25° \times 8 = 90°$); x_8

Answer. $x_8 = 120$ NM

Exercise Group. In Exercise 1.4.7.17–23 we determined the deviation of a wa‘a following a day of sailing (120 nautical miles). Your task now is to calculate the distance the wa‘a has progressed along the reference course (north) for each deviation, denoted as y. Round your answer to the nearest tenth of a nautical mile and remember that one house corresponds to 11.25°.



24. 1 House ($11.25° \times 1 = 11.25°$); y_1

Answer. $y_1 = 117.7$ NM

25. 2 Houses ($11.25° \times 2 = 22.5°$); y_2

Answer. $y_2 = 110.9$ NM

26. 3 Houses ($11.25° \times 3 = 33.75°$); y_3

Answer. $y_3 = 99.8$ NM

27. 4 Houses ($11.25° \times 4 = 45°$); y_4

Answer. $y_4 = 84.5$ NM

28. 5 Houses ($11.25° \times 5 = 56.25°$); y_5

Answer. $y_5 = 66.7$ NM

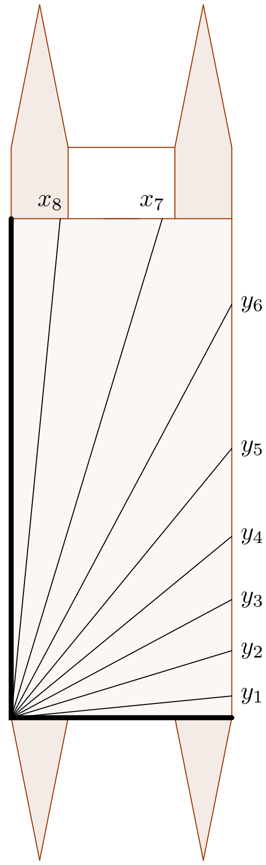
29. 6 Houses ($11.25° \times 6 = 67.5°$); y_6

Answer. $y_6 = 45.9$ NM

30. 7 Houses ($11.25° \times 7 = 78.75°$); y_7

Answer. $y_7 = 23.4$ NM

Exercise Group. One way to determine your bearing on a canoe is by observing and comparing the positions of celestial and other markers relative to your canoe. To facilitate this, you can mark the locations of the Star Compass on the opposite railings from the navigator's seat in the back corner of the canoe. However, since the Star Compass is circular and the canoe is rectangular, accurately placing the markings can be challenging.



When the navigator occupies the port stern (back left) corner of the deck, markers indicating the boundaries between houses can be placed on the corresponding railings on the bow (front) and starboard (right) sides of the canoe. For each value of θ, calculate the distance along the starboard railing (y) or bow railing (x) for a canoe with dimensions $l = 50$ ft and $w = 20$ ft. Round your answers to three decimal places.

31. $\theta = 5.625°$; y_1

Answer. $y_1 = 1.970$ ft

32. $\theta = 16.875°$; y_2

Answer. $y_2 = 6.067$ ft

33. $\theta = 28.125°$; y_3

Answer. $y_3 = 10.69$ ft

34. $\theta = 39.375°$; y_4

Answer. $y_4 = 16.414$ ft

35. $\theta = 50.625°$; y_5

Answer. $y_5 = 24.370$ ft

36. $\theta = 61.875°$; y_6

Answer. $y_6 = 37.417$ ft

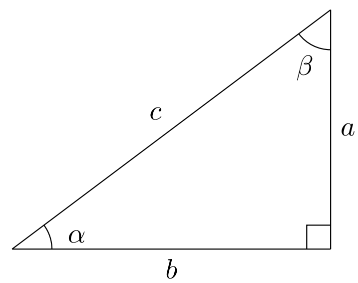
37. $\theta = 73.125°$; x_7

Answer. $x_7 = 15.167$ ft

38. $\theta = 84.375°$; x_8

Answer. $x_8 = 4.925$ ft

Exercise Group. Use the right triangle (not drawn to scale) provided below to solve for the given information. Round your solutions to two decimal places.



39. $a = 5$, $\beta = 35°$. Find b, c, and α

Answer. $b = 3.50$, $c = 6.10$, $\alpha = 55°$

40. $b = 12$, $\beta = 23°$. Find a, c, and α

Answer. $a = 28.27$, $c = 30.71$, $\alpha = 67°$

41. $b = 7$, $\alpha = 75°$. Find a, c, and β

Answer. $a = 26.12$, $c = 27.05$, $\beta = 15°$

42. $c = 4$, $\beta = 50°$. Find a, b, and α

Answer. $a = 3.06$, $b = 2.57$, $\alpha = 40°$

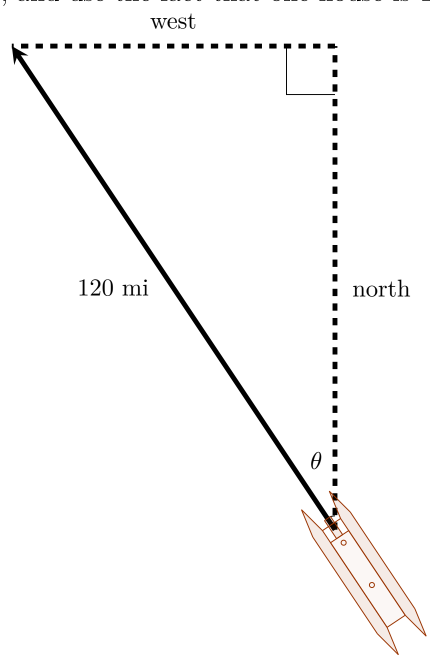
43. $c = 10$, $\alpha = 18°$. Find a, b, and β

Answer. $a = 3.09$, $b = 9.51$, $\beta = 72°$

44. $a = 6$, $\alpha = 38°$. Find b, c, and β

Answer. $b = 7.68$, $c = 9.75$, $\beta = 52°$

45. A wa‘a sails in the direction of the house Nālani Ho‘olua for one day, covering 120 nautical miles. How many nautical miles has the wa‘a traveled north? How many miles has the wa‘a traveled west? To calculate the angle θ, refer to the Star Compass (Figure 1.1.4) to determine the number of houses, and use the fact that one house is 11.25°.

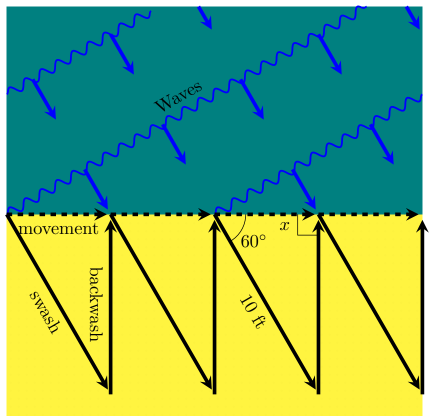


Answer. 66.7 NM west; 99.8 NM north

46. **Movement of Sand.** The movement of sand on a beach is a dynamic process influenced by various factors, such as waves. When waves approach the shore at an angle, they lead to the shifting of sand. During the *swash* phase, as the wave crashes onto the shore, water and sediment move onto the beach following the wave's angle. Subsequently, gravity propels the water and sediment back into the ocean, perpendicular to the shoreline, in a process known as *backwash*. This interplay of swash and backwash creates a zig-zag pattern called *longshore drift*.

Certain beaches undergo seasonal changes in wave direction. Some experience waves from one direction in one season and from another direction in the next, while those predominantly receiving waves from a single direction might accumulate sand in specific areas.

Calculate how far along the shore a single grain of sand moves after a wave breaks at a 60° angle and travels onto the shore for 10 ft before receding back into the ocean.



Answer. 5ft

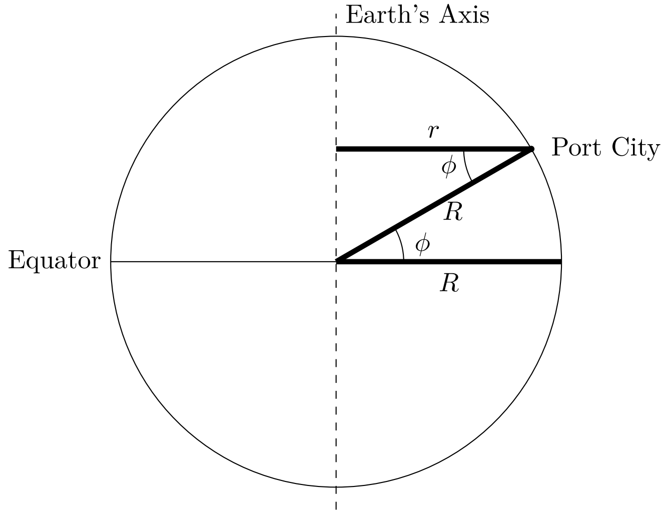
Exercise Group. Between 2013 and 2017, Hōkūleʻa completed a global circumnavigation with a mission *mālama honua* - "care for Island Earth" and to foster a sense of *ʻohana* ("family") for all people and places. This remarkable voyage spanned 40,000 nautical miles and made stops at over 150 ports across 18 nations.

Throughout this voyage, Earth's rotation occurs around an axis that extends from the North Pole to the South Pole. The rotation imparts an angular speed and linear velocity to every point on Earth. Assuming Earth completes one rotation within 24 hours and treating Earth as a perfect sphere with a radius of $R = 4,000$ miles, we can calculate the following parameters for each of the Mālama Honua voyage's ports, given their latitudes (ϕ):

(a) Calculate r, the distance from the port to Earth's Axis of Rotation (in miles, rounded to one decimal place).

(b) Calculate ω, the angular velocity (in radians per hour, rounded to four decimal places).

(c) Calculate v, the linear speed (in miles per hour, rounded to the nearest whole number).



47. Hilo, Hawai'i (19.7216° N)

Answer.

(a) $r = 3,8765.4$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 986$ mi/hr

48. Papeete, Tahiti (17.5325° S)

Answer.

(a) $r = 3,814.2$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 999$ mi/hr

49. Apia, Samoa (13.8507° S)

Answer.

(a) $r = 3,883.7$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,017$ mi/hr

50. Waitangi, Aotearoa (35.2683° S)

Answer.

(a) $r = 32,65.8$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 855$ mi/hr

51. Sydney, Australia (33.8688° S)

Answer.

(a) $r = 3,321.3$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 870$ mi/hr

52. Bali, Indonesia (8.4095° S)

Answer.

(a) $r = 3,957.0$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,036$ mi/hr

53. Port Louis, Mauritius (20.1609° S)

Answer.

(a) $r = 3,754.9$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 983$ mi/hr

54. Cape Town, South Africa (33.9249° S)

Answer.

(a) $r = 3,319.1$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 869$ mi/hr

55. Natal, Brazil (5.7842° S)

Answer.

(a) $r = 3,979.6$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,042$ mi/hr

56. Necker, British Virgin Islands (18.5268° N)

Answer.

(a) $r = 3,792.7$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 993$ mi/hr

57. Yarmouth, Nova Scotia (43.8379° N)

Answer.

(a) $r = 3,792.7$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 755$ mi/hr

58. Balboa, Panama (8.9614°N)

Answer.

(a) $r = 3,951.2$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,034$ mi/hr

59. Galapagos Islands (0.9538° S)

Answer.

(a) $r = 3,999.5$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,047$ mi/hr

60. Rapa Nui (27.1127° S)

Answer.

(a) $r = 3,560.5$ miles;

(b) $\omega = 0.2618$ rad/hr;

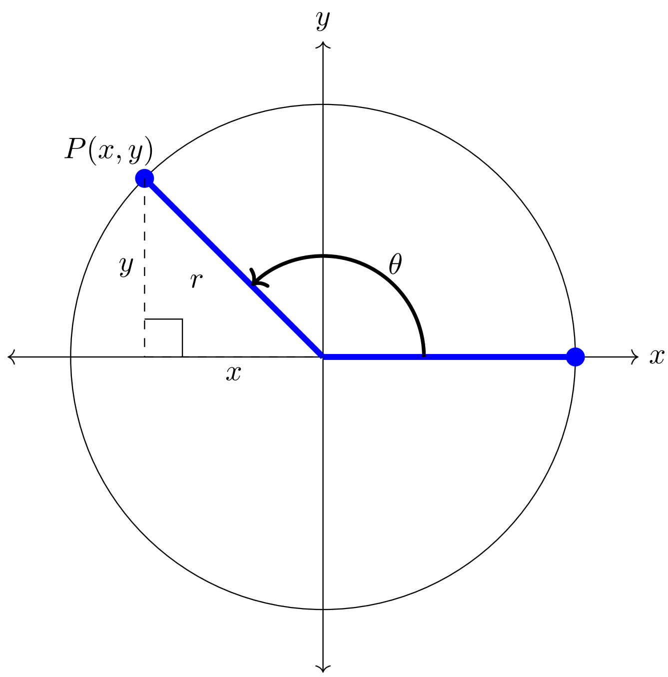
(c) $v = 932$ mi/hr

1.5 Trigonometric Functions of Any Angles

Now that we have been introduced to the six trigonometric functions for special angles in the first quadrant, we can explore their properties across all quadrants.

1.5.1 Determine the Signs of the Trigonometric Functions Based on its Quadrant

Let $P(x, y)$ be a point on the circle. The signs of the six trigonometric functions vary depending on the quadrant in which $P(x, y)$ lies in.



Example 1.5.1 Let $P(x, y)$ is in Quadrant II. Determine the signs for each of the six trigonometric functions.

Solution. Since we are in Quadrant II, $x < 0$ and $y > 0$. Note that $r > 0$. Then we have

$$\sin\theta = \frac{y}{r} = \frac{(+)}{(+)} = (+) \quad \cos\theta = \frac{x}{r} = \frac{(-)}{(+)} = (-) \quad \tan\theta = \frac{y}{x} = \frac{(+)}{(-)} = (-)$$
$$\csc\theta = \frac{r}{y} = \frac{(+)}{(+)} = (+) \quad \sec\theta = \frac{r}{x} = \frac{(+)}{(-)} = (-) \quad \cot\theta = \frac{x}{y} = \frac{(-)}{(+)} = (-)$$

□

You can check the remaining quadrants using a similar approach. Table 1.5.2 and Figure 1.5.3 provide a list of the signs of the six trigonometric functions for each quadrant.

Table 1.5.2 Signs of the trigonometric functions

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

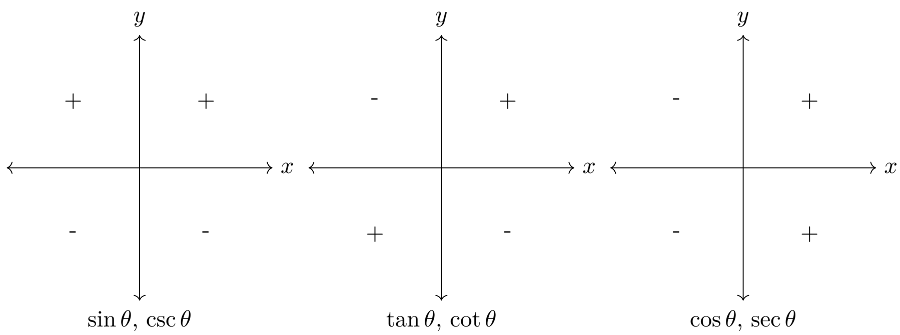


Figure 1.5.3 Signs of trigonometric functions

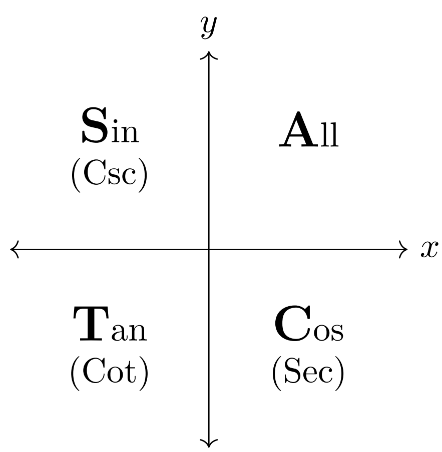
Example 1.5.4 If $\sin\theta < 0$ and $\cos\theta > 0$, what quadrant does θ lie in?

Solution. Since $\sin\theta < 0$, then θ is either in Quadrant III or IV. However, we also have $\cos\theta > 0$ which means that θ is either in Quadrant I or IV. Thus the only quadrant that satisfied both conditions is Quadrant IV. □

Mnemonic devices for remembering the quadrants in which the trigonometric functions are positive are

- "*A* *S*mart *T*rig *C*lass"
- "*A*ll *S*tudents *T*ake *C*alculus"

which correspond to "*A*ll *S*in *T*an *C*os."



Example 1.5.5 Let $\sin\theta = -\frac{12}{13}$ and $\cos\theta = -\frac{5}{13}$. Compute the exact values of the remaining trigonometric functions of θ using identities.

Solution. Since $\sin\theta < 0$ and $\cos\theta < 0$, we refer to Table 1.5.2 and see that θ is in Quadrant III. From Table 1.5.2 we know $\tan\theta > 0$, $\csc\theta < 0$, $\sec\theta < 0$,

$\cot\theta > 0$. From the Quotient Identity (Definition 1.4.3), we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}$$

Next, using the Reciprocal Identities (Definition 1.4.2), we get

$$\begin{aligned}
\csc\theta &= \frac{1}{\sin\theta} = \frac{1}{-\frac{12}{13}} = -\frac{13}{12}\\
\sec\theta &= \frac{1}{\cos\theta} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}\\
\cot\theta &= \frac{1}{\tan\theta} = \frac{1}{\frac{12}{5}} = \frac{5}{12}
\end{aligned}$$

□

1.5.2 Reference Angles

Now that we can determine the signs of trigonometric functions, we will demonstrate how the value of any trigonometric function at any angle can be found from its value in Quadrant I (between 0° and 90° or 0 and $\frac{\pi}{2}$).

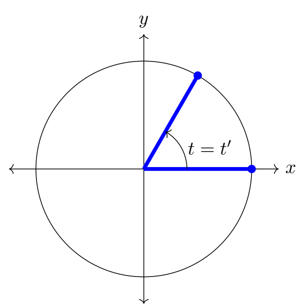
Definition 1.5.6 Let t be a real number. A **reference angle**, t', is the acute angle ($< 90°$) formed by the terminal side of angle t and the x-axis. In other words, it is the shortest distance along the unit circle measured between the terminal side and the x-axis. Angles in Quadrant I are their own reference angles. ♢

Remark 1.5.7 Calculating the reference angle. To calculate the reference angle t‘ for a given angle t:

- In radians, if $t > 2\pi$ or if $t < 0$, add or subtract multiples of 2π to obtain a coterminal angle between 0 and 2π. Then, find the reference angle.
- In degrees, if $t > 360°$ or $t < 0°$, add or subtract multiples of 360° to obtain a coterminal angle between 0° and 360°. Then, find the reference angle.

Quadrant I:

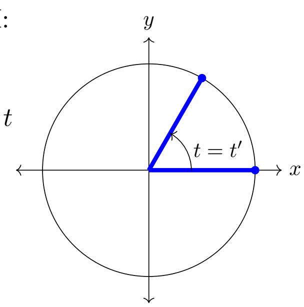
$t' = t$



Quadrant II:

$t' = \pi - t$

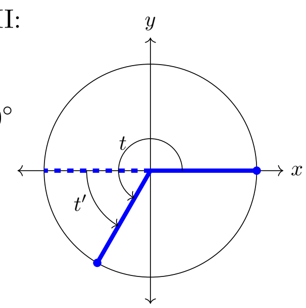
$t' = 180° - t$



Quadrant III:

$t' = t - \pi$

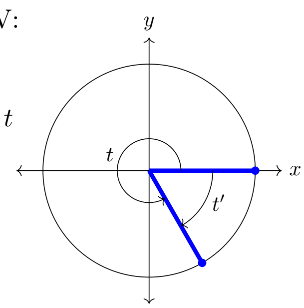
$t' = t - 180°$



Quadrant IV:

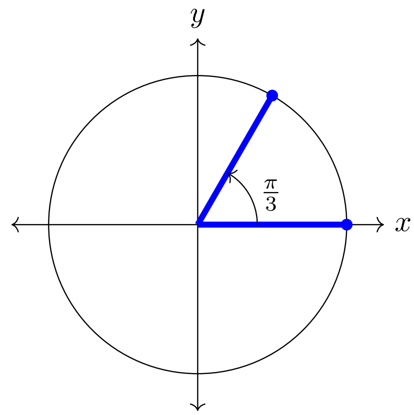
$t' = 2\pi - t$

$t' = 360° - t$



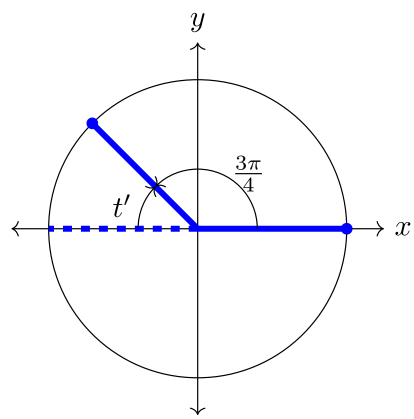
Example 1.5.8 Find the reference angle for each value of t

(a) $t = \frac{\pi}{3}$



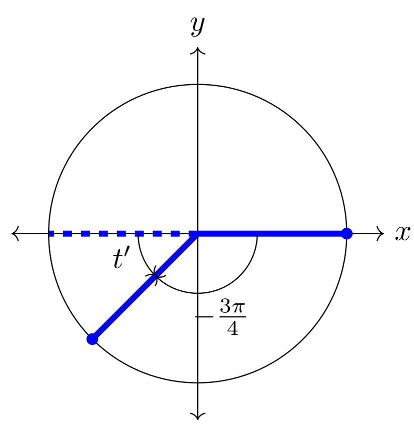
Solution. The angle $t = \frac{\pi}{3}$ is in the first quadrant and so it is its own reference angle: $t = t' = \frac{\pi}{3}$

(b) $t = \frac{3\pi}{4}$



Solution. From the figure, we see that the shortest distance to the x-axis is in the direction of π. We see that $t' + \frac{3\pi}{4} = \pi$ so $t' = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

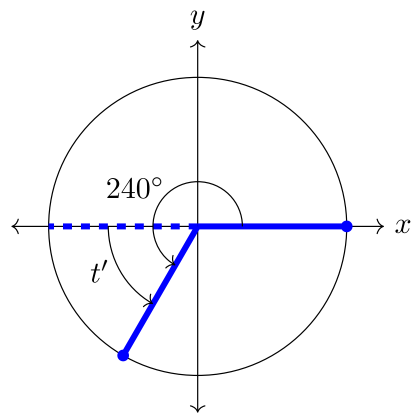
(c) $t = -\frac{3\pi}{4}$



Solution 1. Since $t < 0$, we can add 2π to get $-\frac{3\pi}{4} + 2\pi = \frac{5\pi}{4}$. From the formula, we get $t' = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

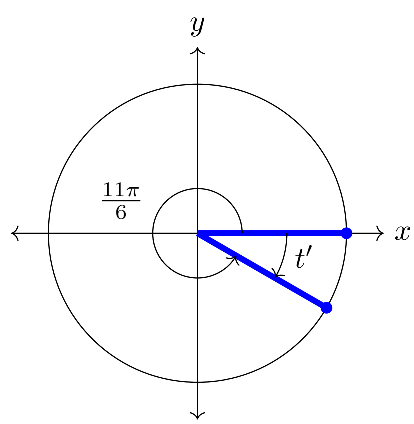
Solution 2. Since $-\frac{3\pi}{4}$ spans only two quadrants counterclockwise, we can treat it similarly to an angle in Quadrant II. By the previous problem, $t' + \frac{3\pi}{4} = \pi$ so $t' = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

(d) $t = 240°$



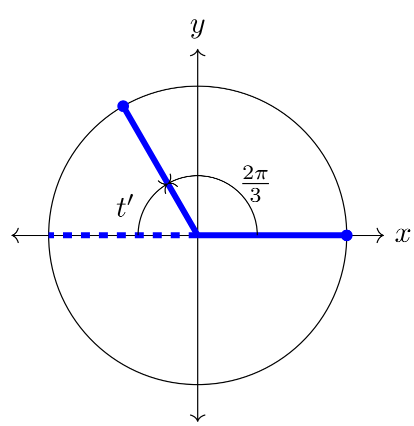
Solution. From the figure we see the shortest distance to the x-axis is towards $180°$. We observe that $240° - t' = 180°$ so $t' = 240° - 180° = 60°$

(e) $t = \frac{11\pi}{6}$



Solution. From the figure, we see the shortest distance to the x-axis is towards 2π. We observe that $t' + \frac{11\pi}{6} = 2\pi$ so $t' = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$.

(f) $t = \frac{2\pi}{3}$



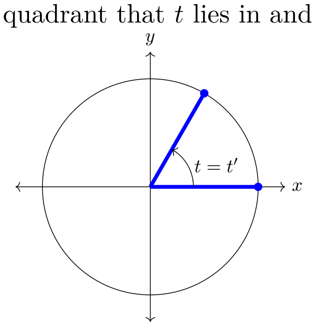
Solution. From the figure, we see the shortest distance to the x-axis is towards π. We observe that $t' + \frac{2\pi}{3} = \pi$ so $t' = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

□

Remark 1.5.9 Calculate an angle in standard position given its quadrant and reference angle. To calculate an angle in standard position, t, given the quadrant that t lies in and the reference angle t',

Quadrant I:

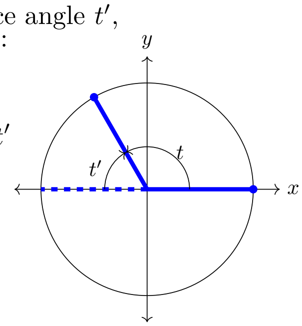
$t = t'$



Quadrant II:

$t = \pi - t'$

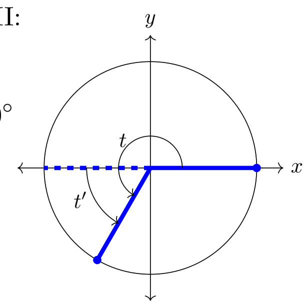
$t = 180° - t'$



Quadrant III:

$t = t' - \pi$

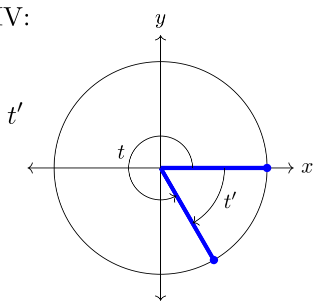
$t = t' - 180°$



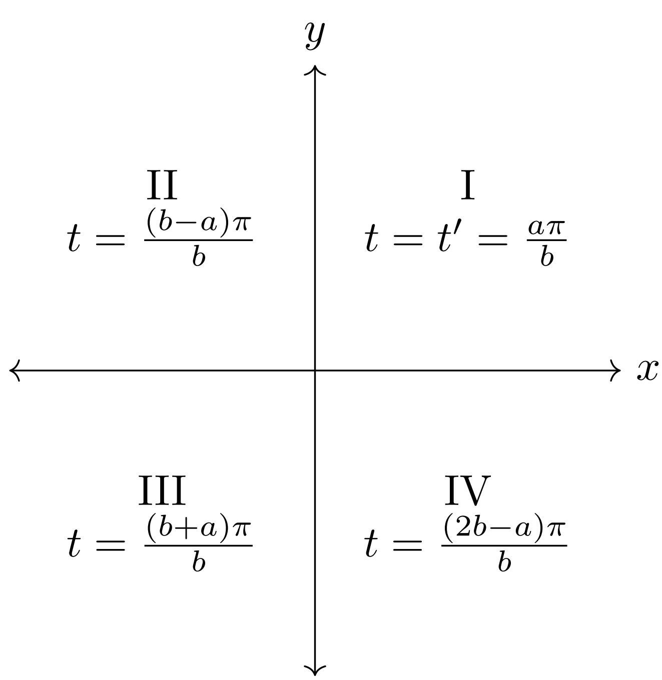
Quadrant IV:

$t = 2\pi - t'$

$t = 360° - t'$

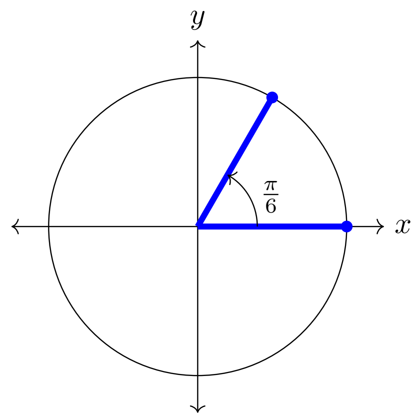


For radians only: If the reference angle (in radians) is of the form $t' = \frac{a\pi}{b}$, then the associated angle in standard position, t, can be calculated by



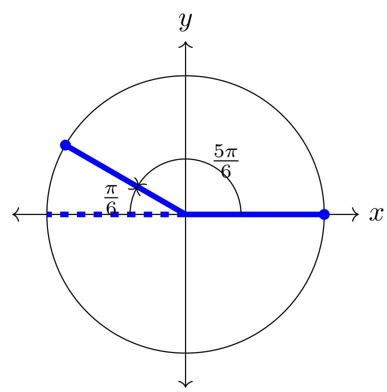
Example 1.5.10 Calculate an angle given its reference angle and quadrant. Given a reference angle, t, compute the associated angle in standard position for Quadrant II, III, and IV.

(a) $t' = \frac{\pi}{6}$



(i) Quadrant II

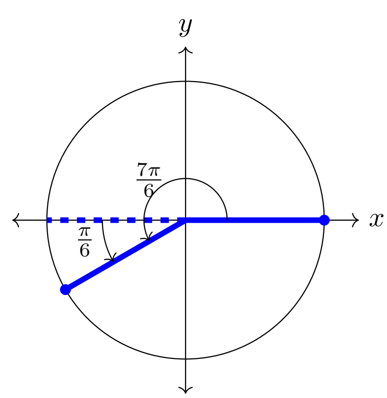
Solution 1. In Quadrant II, the associated angle is $t = \pi - \frac{\pi}{6} = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}$



Solution 2. Since $t' = \frac{\pi}{6} = \frac{1\pi}{6}$, then $t' = \frac{(6-1)\pi}{6} = \frac{5\pi}{6}$

(ii) Quadrant III

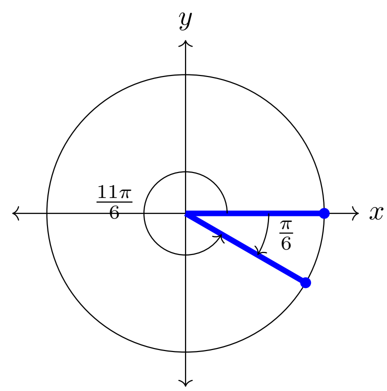
Solution 1. In Quadrant III, the associated angle is $t = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}$



Solution 2. $t' = \frac{(6+1)\pi}{6} = \frac{7\pi}{6}$

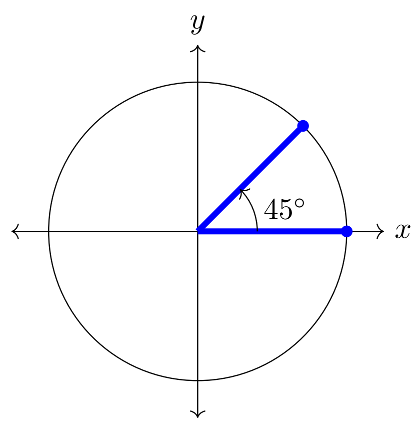
(iii) Quadrant IV

Solution 1. In Quadrant IV, the associated angle is $t = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$



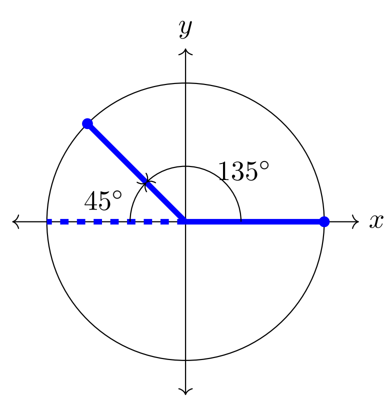
Solution 2. $t' = \frac{(2\cdot 6-1)\pi}{6} = \frac{11\pi}{6}$

(b) $t' = 45°$



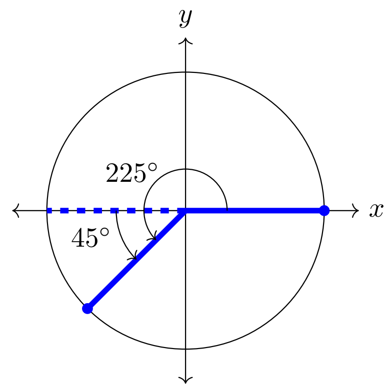
(i) Quadrant II

Solution. In Quadrant II, the associated angle is $t = 180° - 45° = 135°$



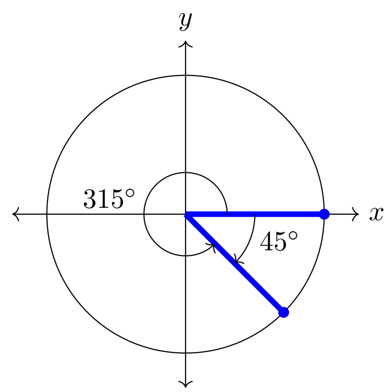
(ii) Quadrant III

Solution. In Quadrant III, the associated angle is $t = 180° + 45° = 225°$



(iii) Quadrant IV

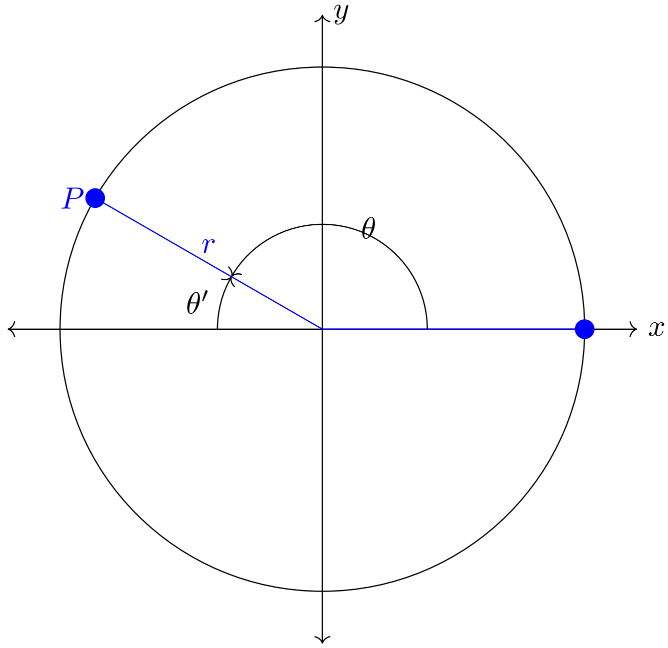
Solution. In Quadrant IV, the associated angle is $t = 360° - 45° = 315°$



□

1.5.3 Evaluating Trigonometric Functions Using Reference Angles

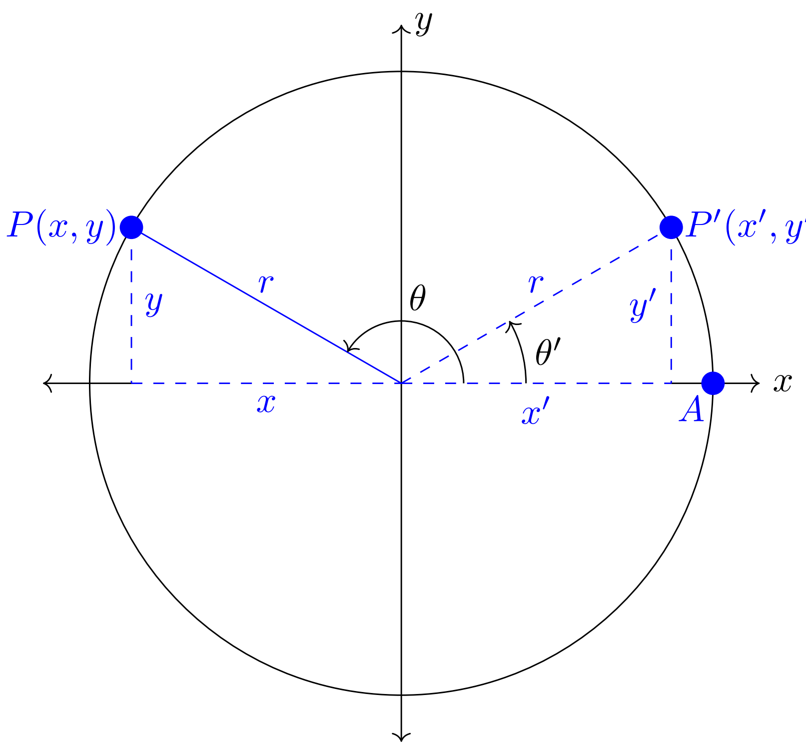
To evaluate trigonometric functions in any quadrant using reference angles, we begin with an angle, θ, that lies in Quadrant II. When evaluating $\sin\theta$ and $\cos\theta$., we begin by plotting θ in standard position and then proceed to determine and draw its corresponding reference angle, θ'.



By definition we know that

$$\sin\theta = \frac{y}{r}; \qquad \cos\theta = \frac{x}{r}$$

Next, we draw the reference angle, θ' in standard position



We now have

$$\sin\theta' = \frac{y'}{r}; \qquad \cos\theta' = \frac{x'}{r}.$$

Notice that the y-coordinates for P and P' share the same value, thus $y = y'$

and we get

$$\sin\theta = \sin\theta'.$$

Similarly, we can see that the x-coordinates of P and P' have opposite values, thus $x = -x'$ and

$$\cos\theta = -\cos\theta'.$$

You may have noticed that we have two similar triangles, differing only in their x-coordinates have opposite values. Consequently, the values of each trigonometric function for the two triangles will match, except for a potential distinction in signs. The sign of each function can be deduced by referring to Table 1.5.2. This approach is applicable across all quadrants. To sum up, we now outline the steps for utilizing reference angles to evaluate trigonometric functions.

Remark 1.5.11 Steps for Evaluating Trigonometric Functions Using Reference Angles. The values of a trigonometric function for a specific angle are equivalent to the values of the same trigonometric function for the reference angle, with a potential distinction in sign. To compute the value of a trigonometric function for any angle, use the following steps

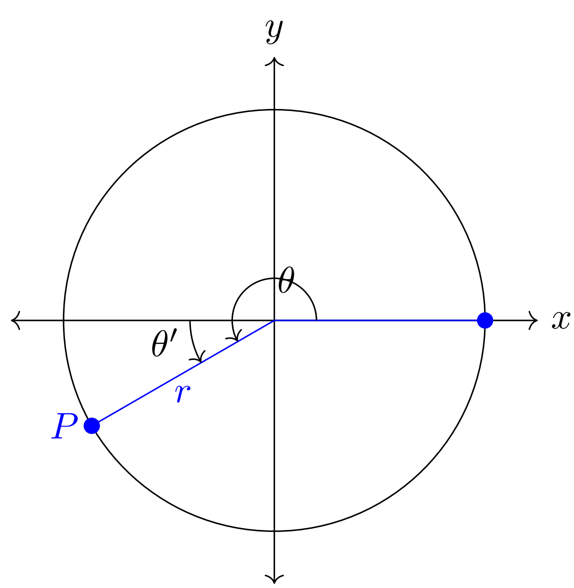
1. Draw the angle in standard position.
2. Determine the reference angle associated with the angle.
3. Evaluate the trigonometric function at the reference angle.
4. Use Table 1.5.2 and the quadrant of the original angle to determine the appropriate sign for the function.

Example 1.5.12 Use the reference angle associated with the given angle to find the exact value of

(a) $\cos 210°$

Solution. We will use the steps for evaluating trigonometric functions using reference angles.

(a) First we draw the angle



(b) The reference angle is

$$\theta' = 210° - 180° = 30°$$

(c) $\cos 30° = \frac{\sqrt{3}}{2}$

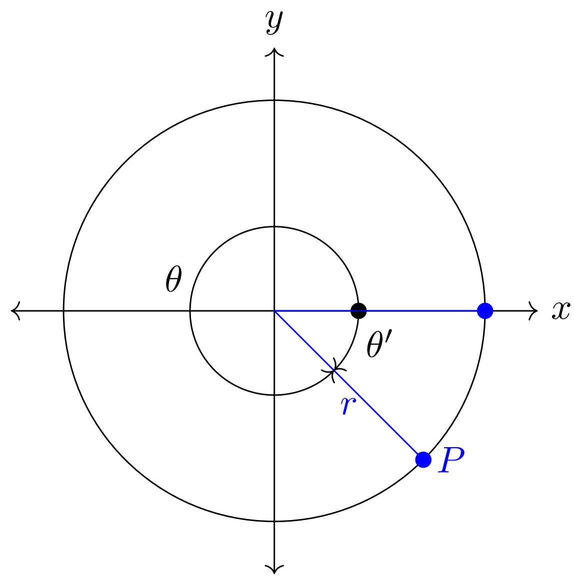
(d) Since 210° lies in Quadrant III, we know that $\cos\theta < 0$, so

$$\cos 210° = -\frac{\sqrt{3}}{2}$$

(b) $\tan\frac{7\pi}{4}$

Solution. We will use the steps for evaluating trigonometric functions using reference angles.

(a) First we draw the angle



(b) The reference angle is

$$2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}$$

(c) $\tan\frac{\pi}{4} = 1$

(d) Since $\frac{7\pi}{4}$ lies in Quadrant IV, we know that $\tan\theta < 0$, so

$$\tan\frac{7\pi}{4} = -1$$

□

Example 1.5.13 Calculate $\sin\theta$ and $\cos\theta$ if $\theta = \frac{20\pi}{3}$

Solution.

1. First we draw the angle

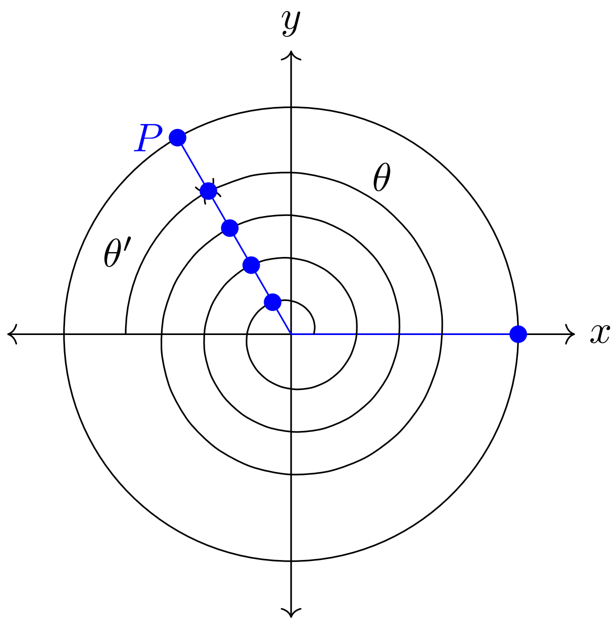


Figure 1.5.14 The angle $\theta = \frac{20\pi}{3}$ makes three rotations before ending in Quadrant II.

2. To obtain the reference angle, we first subtract multiples of 2π from θ to obtain a coterminal angle between 0 and 2π:

$$\begin{aligned} \frac{20\pi}{3} - 2\pi &= \frac{20\pi}{3} - \frac{6\pi}{3} = \frac{14\pi}{3} \\ \frac{20\pi}{3} - 4\pi &= \frac{20\pi}{3} - \frac{12\pi}{3} = \frac{8\pi}{3} \\ \frac{20\pi}{3} - 6\pi &= \frac{20\pi}{3} - \frac{18\pi}{3} = \frac{2\pi}{3} \end{aligned}$$

   From Example 1.5.8, the reference angle for $\frac{2\pi}{3}$ is $\theta' = \frac{\pi}{3}$

3. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$

4. Since $\frac{20\pi}{3}$ lies in Quadrant II, we know that $\sin\theta > 0$ and $\cos\theta < 0$, so

$$\sin\frac{20\pi}{3} = \frac{\sqrt{3}}{2}; \qquad \cos\frac{20\pi}{3} = -\frac{1}{2}$$

□

1.5.4 Periodic Functions

In Figure 1.5.14 of Example 1.5.13, point P corresponds to the angle $\frac{20\pi}{3}$. To determine the reference angle, we subtracted multiples of 2π. Each iteration of 2π retraces the unit circle back to the point P, resulting in a coterminal angle. Therefore

$$\sin\frac{2\pi}{3} = \sin\frac{8\pi}{3} = \sin\frac{14\pi}{3} = \sin\frac{20\pi}{3}$$

Rewriting the angles we get

$$\sin\left(\frac{2\pi}{3} + 0\cdot 2\pi\right) = \sin\left(\frac{2\pi}{3} + 1\cdot 2\pi\right) = \sin\left(\frac{2\pi}{3} + 2\cdot 2\pi\right) = \sin\left(\frac{2\pi}{3} + 3\cdot 2\pi\right)$$

Similarly,

$$\cos\left(\frac{2\pi}{3}+0\cdot 2\pi\right)=\cos\left(\frac{2\pi}{3}+1\cdot 2\pi\right)=\cos\left(\frac{2\pi}{3}+2\cdot 2\pi\right)=\cos\left(\frac{2\pi}{3}+3\cdot 2\pi\right)$$

In general, consider an angle θ measured in radians and its corresponding point P on the unit circle. Adding or subtracting integer multiples of 2π to θ will lead to a point on the unit circle that aligns with P. Thus, the values of sine and cosine for all angles corresponding to point P are equivalent. This leads us to the following periodic properties.

Definition 1.5.15 Periodic Properties.

$$\sin(\theta+2\pi k)=\sin\theta \qquad\qquad \cos(\theta+2\pi k)=\cos\theta$$

where k is any integer. ◊

Functions like these that repeats its values in regular cycles are called *periodic functions*.

Definition 1.5.16 A function f is called **periodic** if there exists a positive number p such that

$$f(\theta+p)=f(\theta)$$

for every θ. The smallest number p is called the **period** of f. ◊

Sine, cosine, cosecant, and secant repeat their values with a period of 2π while tangent and cotangent have a period of π.

Definition 1.5.17 Periodic Properties.

$$\begin{array}{lll} \sin(\theta+2\pi)=\sin\theta & \cos(\theta+2\pi)=\cos\theta & \tan(\theta+\pi)=\tan\theta \\ \csc(\theta+2\pi)=\csc\theta & \sec(\theta+2\pi)=\sec\theta & \cot(\theta+\pi)=\cot\theta \end{array}$$

◊

1.5.5 Trigonometric Table

The Trigonometric Identities and reference angles give us the values of trigonometric functions in Table 1.5.18.

Table 1.5.18 Values of the six trigonometric functions for common angles

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0°$	0	0	1	0	undef	1	undef
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$90°$	$\frac{\pi}{2}$	1	0	undef	1	undef	0
$120°$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
$135°$	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$150°$	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
$180°$	π	0	-1	0	undef	-1	undef
$210°$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$225°$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$240°$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2	$\frac{\sqrt{3}}{3}$
$270°$	$\frac{3\pi}{2}$	-1	0	undef	-1	undef	0
$300°$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2	$-\frac{\sqrt{3}}{3}$
$315°$	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$330°$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$

Remark 1.5.19 Table Made Easy. Table 1.5.18 may seem intimidating but if you recognize the symmetry about 90°, 180°, and 270°, you will only need

to focus on the values for the first quadrant (Table 1.4.6). In fact, you need only produce the values of sine in Quadrant I. Use the Cofunction Identities (Definition 1.4.7) to find the values of cosine. Next, apply the trigonometric identity to find $\tan\theta = \sin\theta/\cos\theta$. Finally, use the the Reciprocal Identities (Definition 1.4.2) to produce $\csc\theta$, $\sec\theta$, and $\cot\theta$.

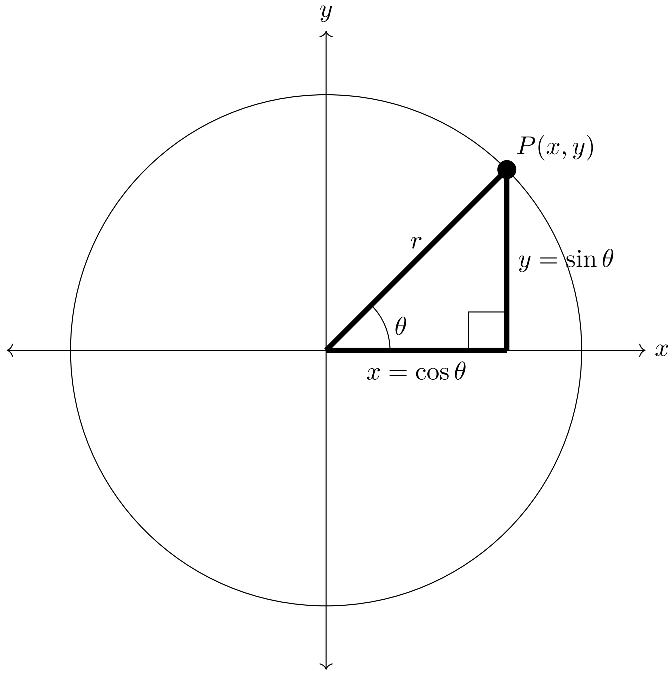
1.5.6 Pythagorean Identities

Definition 1.5.20 Pythagorean Identities.

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

$\Diamond$

Proof. We will use the Pythagorean Theorem to prove the reciprocal identities.



If the point $P(x, y)$ is a point on the circle with radius r, then the formula for the circle is

$$x^2 + y^2 = r^2$$

By definition $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$. Thus we have

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

which is our first Pythagorean Identity. The proofs of the remaining identities are left as exercises. ■

Example 1.5.21 Let θ be an angle in Quadrant IV and let $\cos\theta = \frac{3}{5}$. Calculate the exact values of $\sin\theta$ and $\tan\theta$.

Solution. Substituting our value of $\cos\theta$ into the Pythagorean Identity,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\begin{aligned}\sin^2\theta + \left(\frac{3}{5}\right)^2 &= 1\\ \sin^2\theta + \frac{9}{25} &= 1\\ \sin^2\theta &= 1 - \frac{9}{25}\\ \sin^2\theta &= \frac{16}{25}\end{aligned}$$

Taking the square root of both sides,

$$\sin\theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

Since θ is in Quadrant II, we have $\sin\theta < 0$. Thus we choose the negative answer to get

$$\sin\theta = -\frac{4}{5}$$

Next we use the Quotient Identity to get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{5}\cdot\frac{5}{3} = -\frac{4}{3}$$

□

1.5.7 Even and Odd Trigonometric Functions

Recall that a function f is even if $f(-x) = f(x)$ for all values of x, and a function is odd if $f(-x) = -f(x)$ for all values of x. With this understanding, we can now classify trigonometric functions as either even or odd.

Definition 1.5.22 Even and Odd Trigonometric Properties. The cosine and secant functions are **even**

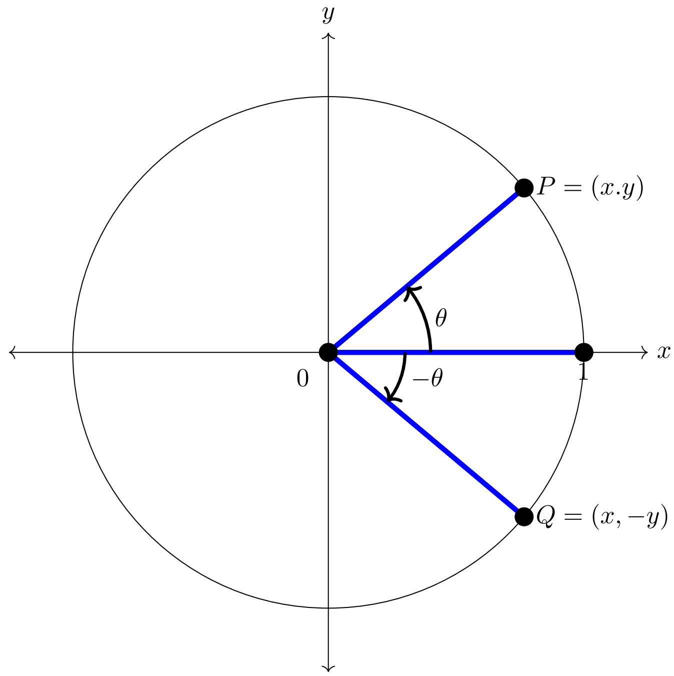
$$\cos(-\theta) = \cos\theta \qquad\qquad \sec(-\theta) = \sec\theta$$

The sine, cosecant, tangent, and cotangent functions are **odd**

$$\begin{aligned}\sin(-\theta) &= -\sin\theta & \csc(-\theta) &= -\csc(\theta)\\ \tan(-\theta) &= -\tan\theta & \cot(-\theta) &= -\cot(\theta)\end{aligned}$$

◊

Proof. Let P be a point on the unit circle corresponding to the angle θ with coordinates (x, y) and Q be the point corresponding to the angle $-\theta$ with coordinates $(x, -y)$.



Using the Definition 1.3.4 for the six trigonometric functions we have

$$\sin\theta = y, \qquad \sin(-\theta) = -y, \qquad \cos\theta = x, \qquad \cos(-\theta) = x$$

So

$$\sin(-\theta) = -y = -\sin\theta, \qquad \cos(-\theta) = x = \cos\theta$$

Thus we conclude that sine is an odd function and cosine is an even function. Next, using the Quotient (Definition 1.4.3) and Reciprocal Identities (Definition 1.4.2) we get

$$\begin{aligned}
\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta,\\
\cot(-\theta) &= \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta,\\
\csc(-\theta) &= \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta,\\
\sec(-\theta) &= \frac{1}{\cos(-\theta)} = \frac{1}{-\cos\theta} = \sec\theta.
\end{aligned}$$

Thus tangent, cotangent, cosecant are odd functions and secant is an even function. ■

Example 1.5.23 Use the even-odd properties of trigonometric functions to determine the exact value of

(a) $\csc(-30^\circ)$

Solution. Since cosecant is an odd function, the cosecant of a negative angle is the opposite sign of the cosecant of the positive angle. Thus, $\csc(-30^\circ) = -\csc 30^\circ = -2$

(b) $\cos(-\theta)$ if $\cos\theta = 0.4$

Solution. Cosine is an even function so $\cos(-\theta) = \cos\theta = 0.4$.

□

1.5.8 Exercises

Exercise Group. Determine the quadrant containing θ given the following

1. $\cot\theta < 0$ and $\cos\theta < 0$

Answer. QII

2. $\csc\theta > 0$ and $\tan\theta > 0$

Answer. QI

3. $\cos\theta > 0$ and $\sin\theta < 0$

Answer. QIV

4. $\sec\theta > 0$ and $\tan\theta > 0$

Answer. QI

5. $\tan\theta < 0$ and $\csc\theta > 0$

Answer. QII

6. $\cot\theta > 0$ and $\sin\theta < 0$

Answer. QIV

7. $\sec\theta < 0$ and $\csc\theta < 0$

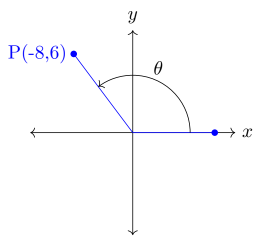
Answer. QIII

8. $\cos\theta < 0$ and $\tan\theta > 0$

Answer. QIII

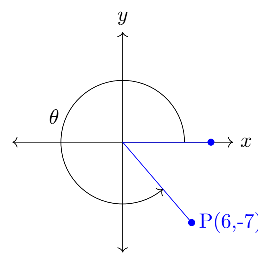
Exercise Group. The point $P(x, y)$ is on the terminal side of angle θ. Determine the exact values of the six trigonometric functions at θ

9.



Answer. $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

10.



Answer. $\sin\theta = -\frac{7\sqrt{85}}{85}$, $\cos\theta = \frac{6\sqrt{85}}{85}$, $\tan\theta = -\frac{7}{6}$, $\csc\theta = -\frac{\sqrt{85}}{7}$, $\sec\theta = \frac{\sqrt{85}}{6}$, $\cot\theta = -\frac{6}{7}$

11. $(3, -4)$

Answer. $\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

12. $(-12, -5)$

Answer. $\sin\theta = -\frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\csc\theta = -\frac{13}{5}$, $\sec\theta = -\frac{13}{12}$, $\cot\theta = \frac{12}{5}$

13. $(-2, -3)$

Answer. $\sin\theta = -\frac{3\sqrt{13}}{13}$, $\cos\theta = -\frac{2\sqrt{13}}{13}$, $\tan\theta = \frac{3}{2}$, $\csc\theta = -\frac{\sqrt{13}}{3}$, $\sec\theta = -\frac{\sqrt{13}}{2}$, $\cot\theta = \frac{2}{3}$

14. $(-4, 4)$

Answer. $\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{\sqrt{2}}{2}$, $\tan\theta = -1$ $\csc\theta = \sqrt{2}$, $\sec\theta = -\sqrt{2}$, $\cot\theta = -1$

15. $(-24, 7)$

Answer. $\sin\theta = \frac{7}{25}$, $\cos\theta = -\frac{24}{25}$, $\tan\theta = -\frac{7}{24}$, $\csc\theta = \frac{25}{7}$, $\sec\theta = -\frac{25}{24}$, $\cot\theta = -\frac{24}{7}$

16. $(9, -40)$

Answer. $\sin\theta = -\frac{40}{41}$, $\cos\theta = \frac{9}{41}$, $\tan\theta = -\frac{40}{9}$, $\csc\theta = -\frac{41}{40}$, $\sec\theta = \frac{41}{9}$, $\cot\theta = -\frac{9}{40}$

Exercise Group. Find the exact value of the remaining five trigonometric functions of θ from the given information.

17. $\tan\theta = -\frac{12}{5}$, θ is Quadrant II

Answer. $\sin\theta = \frac{12}{13}$, $\cos\theta = -\frac{5}{13}$, $\csc\theta = \frac{13}{12}$, $\sec\theta = -\frac{13}{5}$, $\cot\theta = -\frac{5}{12}$

18. $\cos\theta = \frac{3}{5}$, θ is Quadrant IV

Answer. $\sin\theta = -\frac{4}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

19. $\csc\theta = \frac{\sqrt{10}}{2}$, θ is Quadrant II

Answer. $\sin\theta = \frac{\sqrt{10}}{5}$, $\cos\theta = -\frac{\sqrt{15}}{5}$, $\tan\theta = -\frac{\sqrt{6}}{3}$, $\sec\theta = -\frac{\sqrt{15}}{3}$, $\cot\theta = -\frac{6}{2}$

20. $\cos\theta = -\frac{5}{8}$, θ is Quadrant III

Answer. $\sin\theta = -\frac{\sqrt{39}}{8}$, $\tan\theta = \frac{\sqrt{39}}{5}$, $\csc\theta = -\frac{8\sqrt{39}}{39}$, $\sec\theta = -\frac{8}{5}$, $\cot\theta = \frac{5\sqrt{39}}{39}$

21. $\sec\theta = -2$, $\pi < \theta < \frac{3\pi}{2}$

Answer. $\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\csc\theta = -\frac{2\sqrt{3}}{3}$, $\cot\theta = \frac{\sqrt{3}}{3}$

22. $\cot\theta = -\frac{5}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$

Answer. $\sin\theta = -\frac{3\sqrt{34}}{34}$, $\cos\theta = \frac{5\sqrt{34}}{34}$, $\tan\theta = -\frac{3}{5}$, $\csc\theta = -\frac{\sqrt{34}}{3}$, $\sec\theta = \frac{\sqrt{34}}{5}$

23. $\cos\theta = \frac{2}{3}$, $0 < \theta < \pi$

Answer. $\sin\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = -\frac{\sqrt{5}}{2}$, $\csc\theta = \frac{3\sqrt{5}}{5}$, $\sec\theta = -\frac{3}{2}$, $\cot\theta = -\frac{2\sqrt{5}}{5}$

24. $\tan\theta = \frac{7}{4}$, $0 < \theta < \frac{\pi}{2}$

Answer. $\sin\theta = \frac{7\sqrt{65}}{65}$, $\cos\theta = \frac{4\sqrt{65}}{65}$, $\csc\theta = \frac{\sqrt{65}}{7}$, $\sec\theta = \frac{\sqrt{65}}{4}$, $\cot\theta = \frac{4}{7}$

25. $\csc\theta = \frac{3}{2}$, $\tan\theta < 0$

Answer. $\sin\theta = \frac{2}{3}$, $\cos\theta = -\frac{\sqrt{5}}{3}$, $\tan\theta = -\frac{2\sqrt{5}}{5}$, $\sec\theta = -\frac{3\sqrt{5}}{5}$, $\cot\theta = -\frac{\sqrt{5}}{2}$

26. $\sin\theta = \frac{5}{6}$, $\cot\theta > 0$

Answer. $\cos\theta = \frac{\sqrt{11}}{6}$, $\tan\theta = \frac{5\sqrt{11}}{11}$, $\csc\theta = \frac{6}{5}$, $\sec\theta = \frac{6\sqrt{11}}{11}$, $\cot\theta = \frac{\sqrt{11}}{5}$

27. $\sin\theta = -\frac{15}{17}$, $\cos\theta < 0$

Answer. $\cos\theta = -\frac{8}{17}$, $\tan\theta = \frac{15}{8}$, $\csc\theta = -\frac{17}{15}$, $\sec\theta = -\frac{17}{8}$, $\cot\theta = \frac{8}{15}$

28. $\cot\theta = -\frac{1}{3}$, $\sin\theta > 0$

Answer. $\sin\theta = \frac{3\sqrt{10}}{10}$, $\cos\theta = -\frac{\sqrt{10}}{10}$, $\tan\theta = -3$, $\csc\theta = \frac{\sqrt{10}}{3}$, $\sec\theta = -\sqrt{10}$

Exercise Group. Given a reference angle, t', calculate the corresponding angle, t, in standard position, along with the values of $\sin t$, $\cos t$, and $\tan t$ for

(a) Quadrant II

(b) Quadrant III

(c) Quadrant IV

29. $t' = \frac{\pi}{4}$

Answer 1. $t = \frac{3\pi}{4}$, $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$, $\tan\frac{3\pi}{4} = -1$

Answer 2. $t = \frac{5\pi}{4}$, $\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, $\tan\frac{5\pi}{4} = 1$

Answer 3. $t = \frac{7\pi}{4}$, $\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$, $\tan\frac{7\pi}{4} = -1$

30. $t' = \frac{\pi}{3}$

Answer 1. $t = \frac{2\pi}{3}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$

Answer 2. $t = \frac{4\pi}{3}$, $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} = -\frac{1}{2}$, $\tan\frac{4\pi}{3} = \sqrt{3}$

Answer 3. $t = \frac{5\pi}{3}$, $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{5\pi}{3} = \frac{1}{2}$, $\tan\frac{5\pi}{3} = -\sqrt{3}$

31. $t' = 30°$

Answer 1. $t = 150°$,
$\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$,
$\tan 150° = -\frac{\sqrt{3}}{3}$

Answer 2. $t = 210°$,
$\sin 210° = -\frac{1}{2}$,
$\cos 210° = -\frac{\sqrt{3}}{2}$,
$\tan 210° = \frac{\sqrt{3}}{3}$

Answer 3. $t = 330°$,
$\sin 330° = -\frac{1}{2}$, $\cos 330° = \frac{\sqrt{3}}{2}$,
$\tan 330° = -\frac{\sqrt{3}}{3}$

32. $t' = 60°$

Answer 1. $t = 120°$,
$\sin 120° = \frac{\sqrt{3}}{2}$, $\cos 120° = -\frac{1}{2}$,
$\tan 120° = -\sqrt{3}$

Answer 2. $t = 240°$,
$\sin 240° = -\frac{\sqrt{3}}{2}$,
$\cos 240° = -\frac{1}{2}$, $\tan 240° = \sqrt{3}$

Answer 3. $t = 300°$,
$\sin 300° = -\frac{\sqrt{3}}{2}$, $\cos 300° = \frac{1}{2}$,
$\tan 300° = -\sqrt{3}$

Exercise Group. For each angle θ,

(a) Determine the quadrant in which θ lies.

(b) Calculate the reference angle θ'

(c) Use the reference angle, θ' to evaluate the exact values of the six trigonometric functions for θ

33. $\theta = -\frac{3\pi}{4}$

Answer 1. QIII

Answer 2. $\theta' = \frac{\pi}{4}$

Answer 3. $\sin\theta = -\frac{\sqrt{2}}{2}$,
$\cos\theta = -\frac{\sqrt{2}}{2}$,
$\tan\theta = 1$,
$\csc\theta = -\sqrt{2}$,
$\sec\theta = -\sqrt{2}$,
$\cot\theta = 1$

34. $\theta = \frac{4\pi}{3}$

Answer 1. QIII

Answer 2. $\theta' = \frac{\pi}{3}$

Answer 3. $\sin\theta = -\frac{\sqrt{3}}{2}$,
$\cos\theta = -\frac{1}{2}$,
$\tan\theta = \sqrt{3}$,
$\csc\theta = -\frac{2\sqrt{3}}{3}$,
$\sec\theta = -2$,
$\cot\theta = \frac{\sqrt{3}}{3}$

35. $\theta = \frac{11\pi}{6}$

Answer 1. QIV

Answer 2. $\theta' = \frac{\pi}{6}$

Answer 3. $\sin\theta = -\frac{1}{2}$,
$\cos\theta = \frac{\sqrt{3}}{2}$,
$\tan\theta = -\frac{\sqrt{3}}{3}$,
$\csc\theta = -2$,
$\sec\theta = \frac{2\sqrt{3}}{3}$,
$\cot\theta = -\sqrt{3}$

36. $\theta = \frac{7\pi}{3}$

Answer 1. QI

Answer 2. $\theta' = \frac{\pi}{3}$

Answer 3. $\sin\theta = \frac{\sqrt{3}}{2}$,
$\cos\theta = \frac{1}{2}$,
$\tan\theta = \sqrt{3}$,
$\csc\theta = \frac{2\sqrt{3}}{3}$,
$\sec\theta = 2$,
$\cot\theta = \frac{\sqrt{3}}{3}$

37. $\theta = 120°$

Answer 1. QII

Answer 2. $\theta' = 60°$

Answer 3. $\sin\theta = \frac{\sqrt{3}}{2}$,
$\cos\theta = -\frac{1}{2}$,
$\tan\theta = -\sqrt{3}$,
$\csc\theta = \frac{2\sqrt{3}}{3}$,
$\sec\theta = -2$,
$\cot\theta = -\frac{\sqrt{3}}{3}$

38. $\theta = 480°$

Answer 1. QII

Answer 2. $\theta' = 60°$

Answer 3. $\sin\theta = \frac{\sqrt{3}}{2}$,
$\cos\theta = -\frac{1}{2}$,
$\tan\theta = -\sqrt{3}$,
$\csc\theta = \frac{2\sqrt{3}}{3}$,
$\sec\theta = -2$,
$\cot\theta = -\frac{\sqrt{3}}{3}$

Exercise Group. Use the fact that the trigonometric functions are periodic to find the exact value for each expression.

39. $\tan 420°$

Answer. $\sqrt{3}$

40. $\csc 540°$

Answer. Undefined

41. $\sin 765°$

Answer. $\frac{\sqrt{2}}{2}$

42. $\sec 1200°$

Answer. -2

43. $\cot\frac{8\pi}{3}$

Answer. $-\frac{\sqrt{3}}{3}$

44. $\cos\frac{21\pi}{4}$

Answer. $-\frac{\sqrt{2}}{2}$

45. $\tan\frac{35\pi}{6}$

Answer. $\frac{\sqrt{3}}{2}$

46. $\sin\frac{39\pi}{4}$

Answer. $-\frac{\sqrt{2}}{2}$

47. Prove the second Pythagorean Identity (Definition 1.5.20): $1+\tan^2\theta = \sec^2\theta$.

Hint. Begin with $\sin^2\theta+\cos^2\theta = 1$ and divide both sides of the equation by $\cos^2\theta$.

48. Prove the third Pythagorean Identity (Definition 1.5.20): $1+\cot^2\theta+1 = \csc^2\theta$.

Hint. Begin with $\sin^2\theta+\cos^2\theta = 1$ and divide both sides of the equation by $\sin^2\theta$.

Exercise Group. Use the Pythagorean Identity to find the exact value of the following

49. $\sin^2 38°+\cos^2 38°$

Answer. 1

50. $\csc^2 13°-\cot^2 13°$

Answer. 1

51. $\cot^2 200° - \csc^2 200°$

Answer. -1

52. $\sec^2\frac{6\pi}{5} - \tan^2\frac{6\pi}{5}$

Answer. 1

53. $\tan^2\frac{5\pi}{7} - \sec^2\frac{5\pi}{7}$

Answer. -1

54. $\sin^2\frac{14\pi}{13} + \cos^2\frac{14\pi}{13}$

Answer. 1

Exercise Group. Use the Pythagorean Identities to express the first trigonometric function of θ in terms of the second function, given the quadrant.

55. $\sin\theta$, $\cos\theta$, Quadrant III

Answer. $\sin\theta = -\sqrt{1-\cos^2\theta}$

56. $\cos\theta$, $\sin\theta$, Quadrant II

Answer. $\cos\theta = -\sqrt{1-\sin^2\theta}$

57. tan, $\sec\theta$, Quadrant IV

Answer. $\tan\theta = -\sqrt{\sec^2\theta-1}$

58. $\cot\theta$, $\csc\theta$, Quadrant III

Answer. $\cot\theta = \sqrt{\csc^2\theta-1}$

59. $\tan\theta$, $\sin\theta$, Quadrant III

Answer. $\tan\theta = -\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

60. $\tan\theta$, $\cos\theta$, Quadrant II

Answer. $\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$

Exercise Group. Use the Pythagorean Identities to find the exact values of the remaining five trigonometric functions of θ from the given information.

61. $\tan\theta = -\frac{4}{3}$, θ is in Quadrant IV

Answer. $\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

62. $\cos\theta = -\frac{1}{4}$, θ is in Quadrant II

Answer. $\sin\theta = \frac{\sqrt{15}}{4}$, $\tan\theta = -\sqrt{15}$, $\csc\theta = \frac{4\sqrt{15}}{15}$, $\sec\theta = -4$, $\cot\theta = -\frac{\sqrt{15}}{15}$

63. $\sin\theta = -\frac{2}{3}$, θ is in Quadrant III

Answer. $\cos\theta = -\frac{\sqrt{5}}{3}$, $\tan\theta = \frac{2\sqrt{5}}{5}$, $\csc\theta = -\frac{3}{2}$, $\sec\theta = -\frac{3\sqrt{5}}{5}$, $\cot\theta = \frac{\sqrt{5}}{2}$

64. $\cos\theta = \frac{3}{5}$, θ is in Quadrant IV

Answer. $\sin\theta = -\frac{4}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

Exercise Group. Use the even and odd properties to evaluate the following

65. $\cos(-60°)$

Answer. $\frac{1}{2}$

66. $\tan(-225°)$

Answer. -1

67. $\csc(-330°)$

Answer. 2

68. $\sin(-90°)$

Answer. -1

69. $\cot(-300°)$

Answer. $\frac{\sqrt{3}}{3}$

70. $\sec(-150°)$

Answer. $-\frac{2\sqrt{3}}{3}$

71. $\sin\left(-\frac{11\pi}{6}\right)$

Answer. $\frac{1}{2}$

72. $\tan\left(-\frac{5\pi}{4}\right)$

Answer. -1

73. $\cos\left(-\frac{4\pi}{3}\right)$

Answer. $-\frac{1}{2}$

74. $\tan(-\pi)$

Answer. 0

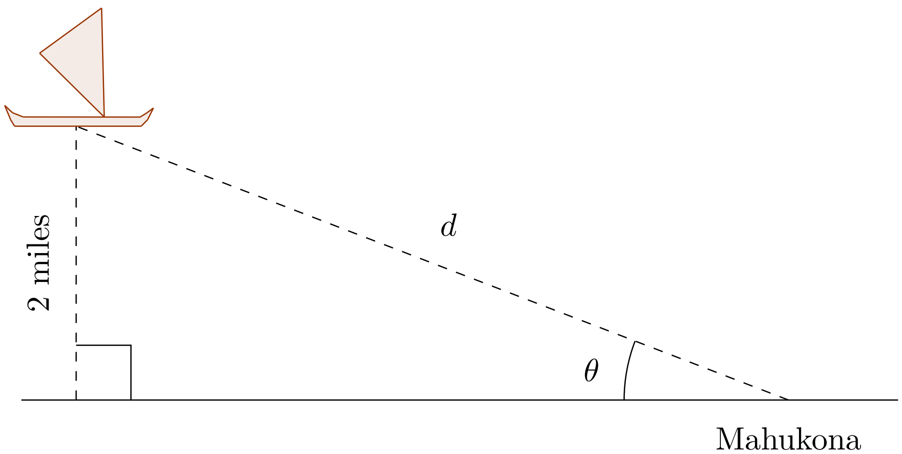
75. $\sec\left(-\frac{\pi}{4}\right)$

Answer. $\sqrt{2}$

76. $\csc\left(-\frac{7\pi}{6}\right)$

Answer. 2

Exercise Group. The Makali‘i is sailing along the Kohala Coast, maintaining a distance of two nautical miles from the shore. An observer at Mahukona is monitoring Makali‘i’s passage. Let d denote the length of the line connecting Makali‘i to the Mahukona observer. Given θ as the angle formed between d and the shore, determine Makali‘i’s distance, d, from the observer for each value of θ, rounded to one decimal place.



77. $\theta = 30°$

Answer. 4 NM

78. $\theta = 45°$

Answer. 2.8 NM

79. $\theta = 60°$

Answer. 2.3 NM

80. $\theta = 90°$

Answer. 2 NM

81. $\theta = 120°$

Answer. 2.3 NM

82. $\theta = 135°$

Answer. 2.8 NM

83. $\theta = 150°$

Answer. 4 NM

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