Trigonometry Through Wayfinding and Navigation Across the Pacific

Trigonometry Through Wayfinding and Navigation Across the Pacific

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For my family

Acknowledgements

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Preface

About this book: This book integrates navigation into trigonometry curriculum.

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Chapters 1-3 are available to down as a PDF at https://www.kamuelayong.com/Trigonometry-Yong-Chapters-1-3.pdf

Core Standards

The curriculum in this book aligns with the Common Core State Standards: thecorestandards.org[3]

- High School: Algebra
  - Creating Equations.[4]
    - Create equations that describe numbers or relationships.
      - HSA.CED.A.2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.(Section 2.3)
- High School: Functions
  - Interpreting Functions[5]
    - Analyze functions using different representations.
      - HSF.IF.C.7: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. (Section 2.1, Section 2.2)
      - HSF.IF.C.7.e: Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. (Section 2.1, Section 2.2)
  - Building Functions.[6]
    - Build a function that models a relationship between two quantities.
      - HSF.BF.A.1: Write a function that describes a relationship between two quantities. (Section 2.3)
    - Build new functions from existing functions.
      - HSF.BF.B.3: Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (Subsection 2.1.4, Subsection 2.2.9)

[3] `http://www.thecorestandards.org/`
[4] `https://www.thecorestandards.org/Math/Content/HSA/CED/`
[5] `https://www.thecorestandards.org/Math/Content/HSF/IF/`
[6] `https://www.thecorestandards.org/Math/Content/HSF/BF/`

- Trigonometric Functions.[7]
  - Extend the domain of trigonometric functions using the unit circle.
    - HSF.TF.A.1: Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. (Definition 1.2.19)
    - HSF.TF.A.2: Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (Subsection 1.3.2)
    - HSF.TF.A.3: Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for x, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number. (Subsection 1.4.2, Subsection 1.5.3)
    - HSF.TF.A.4: Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions. Model periodic phenomena with trigonometric functions. (Subsection 1.5.4, Subsection 1.5.7)
  - Model periodic phenomena with trigonometric functions.
    - HSF.TF.B.5: Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. (Subsection 2.1.4, Section 2.3)
    - HSF.TF.B.6: Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. (Subsection 2.4.1)
    - HSF.TF.B.7: Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. (Exercise 2.1.5.46, Exercise Group 2.1.5.47–50, Exercise Group 2.2.10.37–42, Exercise Group 2.2.10.43–52, Example 2.4.18, Example 2.4.19, Example 2.4.17)
  - Prove and apply trigonometric identities.
    - HSF.TF.C.8: Prove the Pythagorean identity $\sin^2\theta+\cos^2\theta = 1$ and use it to find $\sin\theta$, $\cos\theta$, or $\tan\theta$ given $\sin\theta$, $\cos\theta$, or $\tan\theta$ and the quadrant of the angle. (Definition 1.5.20)
    - HSF.TF.C.9: Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems. (Proof 3.2.1.1, Example 3.2.10, Proof 3.2.3.1)

- High School: Geometry
  - Similarity, Right Triangles, and Trigonometry Define trigonometric ratios and solve problems involving right triangles.[8]
    - Define trigonometric ratios and solve problems involving right triangles.

[7]https://www.thecorestandards.org/Math/Content/HSF/TF/
[8]https://www.thecorestandards.org/Math/Content/HSG/SRT/

- HSG.SRT.C.6: Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (Definition 1.4.1)
- HSG.SRT.C.7: Explain and use the relationship between the sine and cosine of complementary angles. (Definition 1.4.7, Remark 1.4.9)
- HSG.SRT.C.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. (Subsection 1.4.6)

■ Apply trigonometry to general triangles

- HSG.SRT.D.9: Derive the formula Area $= \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. (Definition 4.1.11)
- HSG.SRT.D.9: Prove the Laws of Sines and Cosines and use them to solve problems. (Subsection 4.1.1, Subsection 4.2.1)
- HSG.SRT.D.9: Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

◦ Circles.[9]

■ Find arc lengths and areas of sectors of circles

- HSG.C.B.5: Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. (Definition 1.2.19, Theorem 1.2.26, Definition 1.2.29)

[9] `https://www.thecorestandards.org/Math/Content/HSG/C/`

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Chapter 1

Trigonometric Functions

1.1 Pacific Island Navigation

For centuries, voyagers have navigated vast distances across the Pacific, long before the use of magnetic compasses or modern instruments. They relied on observations of celestial bodies, such as stars and the sun, as well as natural elements like ocean patterns, winds, bird behaviors, and other environmental cues to determine their position relative to known landmarks such as islands, reefs, and continents. Over time, much of this traditional navigational knowledge was lost in many parts of the Pacific. However, some islands, particularly in Micronesia and on Taumako Island in the Solomon Islands, managed to preserve the art and science of traditional navigation. These places continued to uphold the practice, teaching new generations to build ocean-going canoes and develop navigational skills based on profound knowledge of the natural world.

1.1.1 Micronesia

Navigators in Micronesia utilize the *paafu* mat or map (shown in Figure 1.1.1). It is often misunderstood and misinterpreted as a Star "Compass" due to its use of stars and constellations for direction finding. However, paafu serves a different purpose and is not equivalent to the cardinal directionality marked by compasses (North, South, East, and West). Instead, it is a learning and teaching tool designed to teach the locational positions of islands, locales, or canoes relative to other places. This is achieved by observing the rising and setting points of stars and constellations, which act as markers for different locations.

Figure 1.1.1 Paafu mat or map - Photo courtesy of Kānehūnāmoku Voyaging Academy.

A constellation is a cluster of stars whose shapes and meanings reflect and carry cultural significance. In modern society, a well-known cluster of stars in the southern hemisphere, shaped like a cruciform, is commonly referred to as the "Southern Cross." However, for the people of Polowat and others from the Central Carolines region in the western Pacific, this same constellation resembles the triggerfish and is named accordingly.

In the Central Carolines, the general location in the celestial sky where stars appear to rise after sundown is referred to as "*tan*." This term is often mistakenly translated as "east" due to the modern association of stars (like the sun) with "rising" in the east. However, it is important to note that "tan" means "rising" and not "eastward."

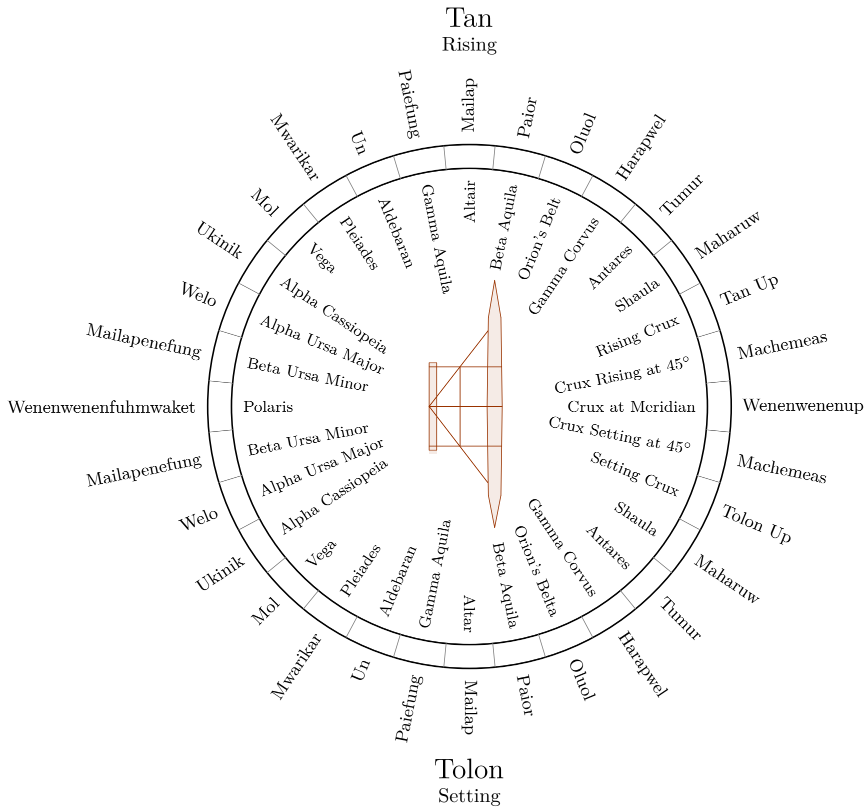


Figure 1.1.2 The paafu, or Micronesian Star Compass. Stars are identified using the Polowat dialect of the Chuukese language as it is used by members of the Weriyeng School of navigation.

Figure 1.1.2 orients the cardinal direction known as "east" at the top of the page, and so the top half of this diagram is also identified as "tan" – where stars appear to rise. The diagram illustrates the apparent path of stars across the sky each night and day (though most stars are not visible during the day) and throughout a year. In this system, the rising and presence of specific stars mark months, and these same stars will eventually set at a point horizontally opposite to where they rose. This point corresponds to the cardinal direction known as "west," which is referred to as "*tolon*" – the area where the stars "set" or go down.

In the paafu "map" shown in Figure 1.1.2, a canoe is placed at the center, and the star called Mailap (Altair) marks due east. The time and location when Mailap rises are referred to as "Tan Mailap" (rising Mailap), while the time and place it sets are referred to as "Tolon Mailap" (setting Mailap). The map's orientation places east, or the rising points of the navigation or paafu stars, at the top of the circle, and west, or the setting points of these stars, at the bottom of the circle. As a map, paafu uses the rising and setting points of stars to mark places around a given locale, which is placed at the center of the circle. Table 1.1.3 displays the star and constellation names, provided in both Polowat and according to the International Astronomical Union, listed in the order of their rising during the third week of March in Polowat.

Table 1.1.3 Star and constellation names in Polowat.

Polowat	International Astronomical Union
Wenenwenenfuhmwaket	Polaris (always above the horizon)
Tan Mwarikar [Mahrah-ker]	Pleiades aka Seven Sisters
Tan Un [Oon]	Aldebaran
Tan Uliul [Ooh-lee-ool]	Orion's Belt
Tan Harapwel [Ah-rah-pwol]	Gamma Corvus
Tan Mailapenefang [My Lap in a Fang]	Beta Ursa Minor in Big Dipper
Tan Up [Oop]	Crux or Southern Cross at Rising
Machemeas [Matche-may-ess]	Crux or S. Cross at 45° [1]
Tan Welo [Well-Ah]	Alpha Ursa Major in Big Dipper
Wenenwenenup [Wehneh wehnen Oop]	Crux or S. Cross at Meridian or upright
Tan Tumur [Two More]	Antares or Scorpio's tail
Tan Maharuw [Maa-Haa-Roo]	Shaula or Scorpio's stinger
Tan Mol [Mohl]	Vega
Tan Mailap [My Lap]	Altair
Tan Paiefung [Pie Efung]	Gamma Aquila
Tan Paior [Pie Or]	Beta Aquila
Tan Ukinik [Icky Nick]	Cassiopeia

Paafu can also be used to identify the direction in which moving objects, such as canoes, or creatures such as birds, fish, and humans, are heading or coming from. This version of paafu utilizes the Polowat dialect of the Chuukese language, as used by members of the Weriyeng School of navigation.

1.1.2 Hawaiʻi

With the aim of reviving wayfinding in Hawaiʻi, Nainoa Thompson journeyed to the island of Satawal in the Federated States of Micronesia to learn from master navigator Mau Piailug, affectionately known as Papa Mau. Using this knowledge, Thompson adopted the paafu method, leading to the creation of the Hawaiian Star Compass, also referred to as the *Kūkuluokalani* (Figure 1.1.4).

In the star compass, featuring the figure of an ʻiwa or great frigatebird at its center, Thompson divides the visual horizon into 32 equidistant points around a circle, referred to as houses. Each house in the Hawaiian Star Compass represents a distinct segment of the horizon—an arc of 11.25°—where celestial bodies such as the sun, stars, moon, and planets rise and set. In the same way that we use addresses to locate homes, each celestial body has its own address represented by these houses.

[1] The "Tan" prefix is not used for this position, because it is no longer rising.

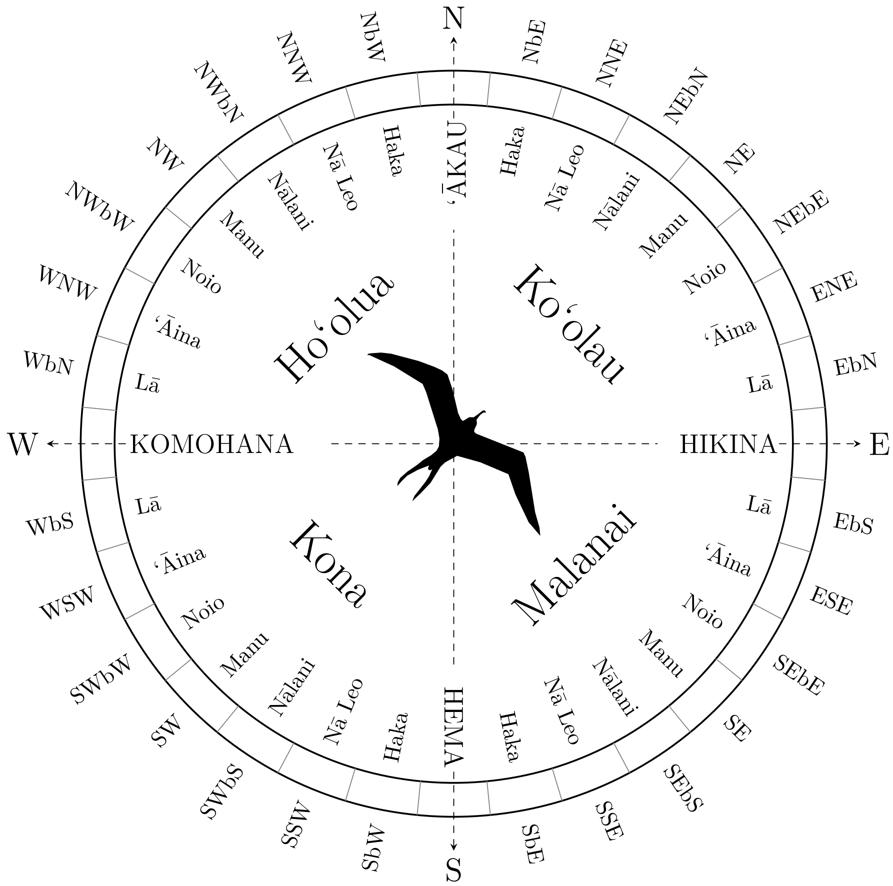


Figure 1.1.4 Hawaiian Star Compass, also known as the Kūkuluokalani.

The four cardinal points align with particular houses. Stars rise from the horizon called *Hikina* (“To Arrive”) or East and set on the horizon called *Komohana* (“To Enter”) or West. If you face Komohana (West) with your back towards Hikina (East), your right will point towards *ʻĀkau* (“Right”) or North, and your left will point towards *Hema* (“Left”) or South. The Hawaiian Star Compass is oriented with North at the top.

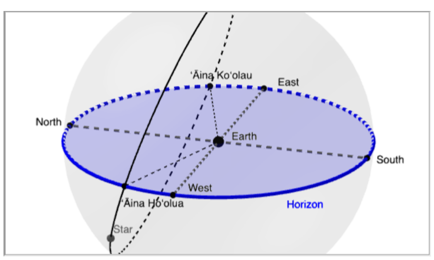
The star compass is divided into four quadrants, each named after winds in Hawaiʻi. *Koʻolau* is the Northeast quadrant named for the trade winds; *Hoʻolua* is the Northwest quadrant, *Kona* is the Southwest quadrant; and *Malanai* is the Southeast quadrant.

Each house on the star compass is given a name. The corresponding houses in the east and west share the same name. Starting from the east or west and moving northwards and southwards, the first house on either side of Hikina (East) and Komohana (West) is called *Lā* (Sun). It is followed by *ʻĀina* (Land), *Noio* (Tern), *Manu* (Bird), *Nālani* (Heavens), *Nā Leo* (Voices), and *Haka* (Empty). The 32 houses in the Hawaiian Star Compass correspond to the points of the 32-wind compass rose (Table 1.1.5).

Table 1.1.5 The houses of the Hawaiian Star Compass and the corresponding points on the 32-wind compass rose.

Star Compass	32 Point Compass		Star Compass	32 Point Compass	
House	Symbol	Name	House	Symbol	Name
Hikina	E	East	Komohana	W	West
Lā Koʻolau	EbN	East by North	Lā Kona	WbS	West by South
ʻĀina Koʻolau	ENE	East-northeast	ʻĀina Kona	WSW	West-southwest
Noio Koʻolau	NEbE	Northeast by East	Noio Kona	SWbW	Southwest by West
Manu Koʻolau	NE	Northeast	Manu Kona	SW	Southwest
Nālani Koʻolau	NEbN	Northeast by North	Nālani Kona	SWbS	Southwest by South
Nā Leo Koʻolau	NNE	North-northeast	Nā Leo Kona	SSW	South-southwest
Haka Koʻolau	NbE	North by east	Haka Kona	SbW	South by West
ʻĀkau	N	North	Hema	S	South
Haka Hoʻolua	NbW	North by West	Haka Malanai	SbE	South by East
Nā Leo Hoʻolua	NNW	North-northwest	Nā Leo Malanai	SSE	South-southeast
Nālani Hoʻolua	NWbN	Northwest by North	Nālani Malanai	SEbS	Southeast by South
Manu Hoʻolua	NW	Northwest	Manu Malanai	SE	Southeast
Noio Hoʻolua	NWbW	Northwest by West	Noio Malanai	SEbE	Southeast by East
ʻĀina Hoʻolua	WNW	West-northwest	ʻĀina Malanai	ESE	East-southeast
Lā Hoʻolua	WbN	West by North	Lā Malanai	EbS	East by South

Celestial bodies move along parallel paths across the sky from East to West, rising and setting in the same house, remaining in the same hemisphere. For example, if a star arrives in the Koʻolau (northeast) quadrant in the star house ʻĀina, it will arc overhead, staying in the northern hemisphere, and enter the horizon in the same house it arrived in, ʻĀina, but in the Hoʻolua (northwest) quadrant (see Figure 1.1.6). Similarly, if a star arrives in the Malanai (southeast) quadrant in the house Lā, it will remain in the southern hemisphere as it arcs overhead and enters the horizon in the house Lā in the Kona (southwest) quadrant.



Standalone

Figure 1.1.6 In the celestial sphere, stars rise in the east, arc across the sky, and set in the west. Each star will both rise and set in the same house. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

The star compass also serves as a guide for determining direction based on wind and ocean swells. As the wind and swells move, they intersect the star compass diagonally. For example, if a wind blows from the house Noio in the Koʻolau (northeast) quadrant, it will blow in the direction of the Kona (southwest) quadrant and eventually exit in the same house, Noio.

Observations play a key role in determining direction using the star compass. At night, Thompson relies on approximately 220 stars, memorizing where they rise and set on the horizon to navigate. During the day, we can use the sun's position on the horizon to gauge direction, but this method is only effective when the sun is near the horizon at sunrise and sunset. Alternatively, one can memorize the wind and wave directions, checking for any changes between sunrise and sunset to establish their current direction.

The canoe itself can serve as a compass, as shown in Figure 1.1.7. From the navigator's seat on either corner of the stern (back) of the deck, we can observe features such as the rising sun and mark their positions on the Star Compass located on the canoe. It is important to note that the locations of the houses on this Star Compass are in relation to the canoe, not to a fixed map. For instance, only when the canoe is pointed towards the north will Hikina (East) be the house to the right. Depending on the canoe's orientation, at other times, Hikina may appear several houses further up the deck.

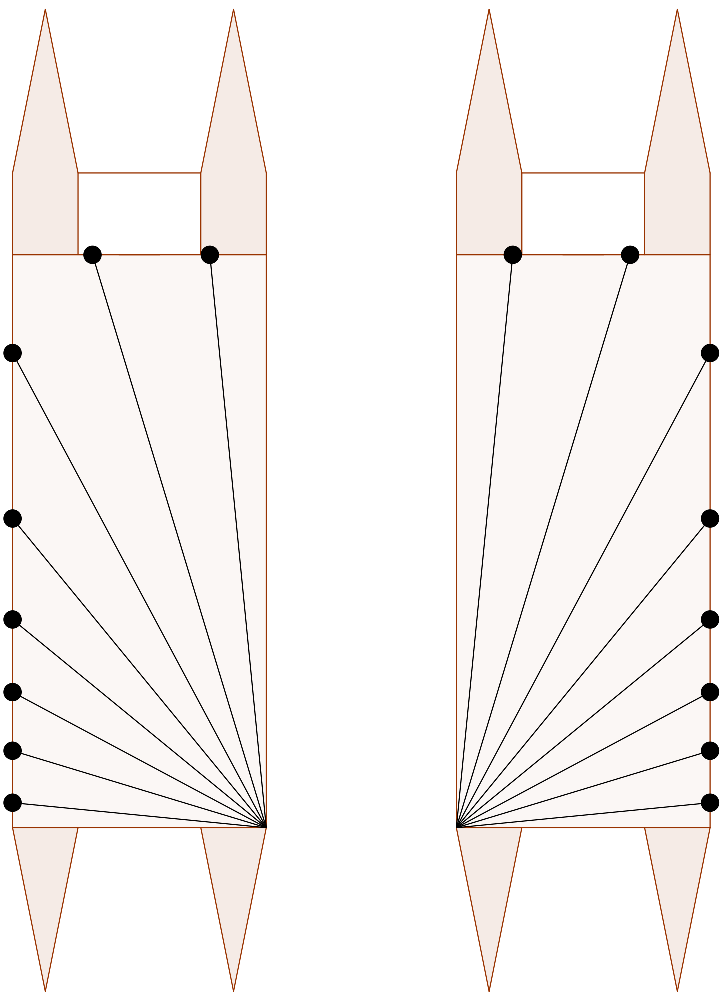


Figure 1.1.7 The deck of a canoe can be used as a compass to help crew and navigators.

1.1.3 Marshall Islands

The Marshallese people use stick charts as navigational tools. These stick charts are constructed using a lattice-like structure made from curved and straight sticks, typically formed by tying together the midribs of coconut fronds. The curved sticks represent the islands and how they bend and refract the ocean swells, while the straight sticks symbolize the major wave patterns in the surrounding waters.

The shells placed on the sticks indicate the relative locations of islands within the Marshall Islands archipelago. These shells serve as markers, helping navigators remember the positions of specific islands along their voyages.

Each stick chart is unique to its creator, reflecting their individual knowledge and experiences. The personalization of the stick charts allows navigators to develop a deep understanding of the ocean currents, wave patterns, and island locations in their specific region.

The stick charts serve as mental maps or navigational aids, allowing experienced navigators to visualize and recall the complex information while on their journeys. Navigators would memorize the stick charts, internalizing the knowledge embedded within them, enabling them to navigate the open ocean.

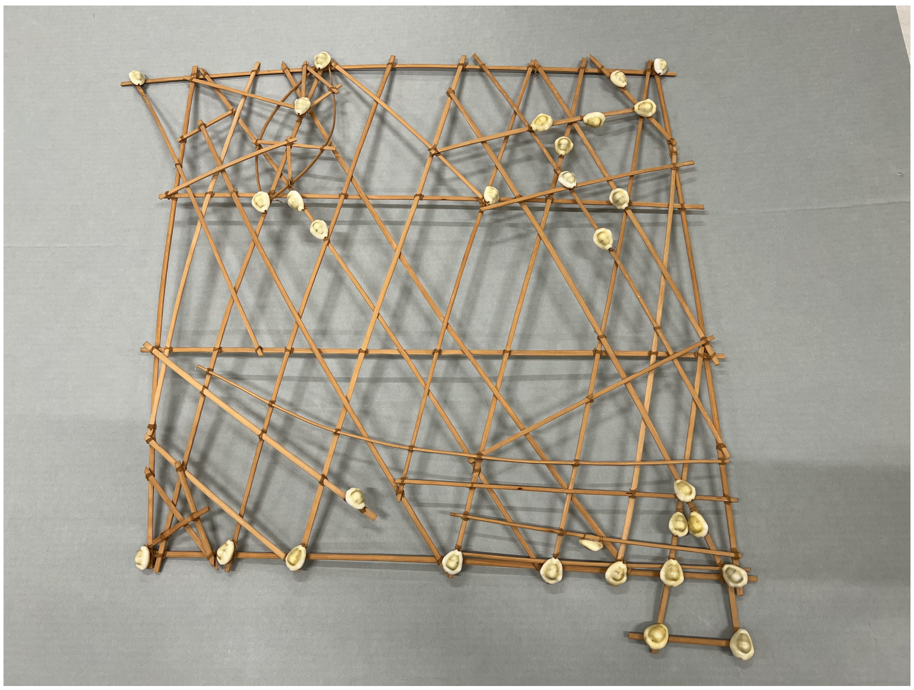
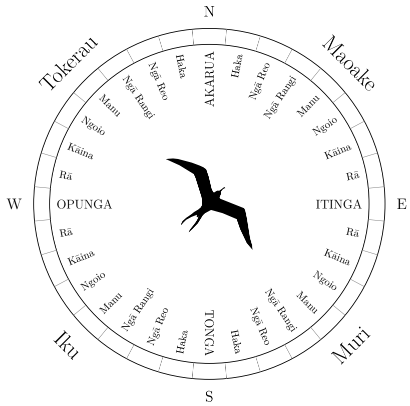


Figure 1.1.8 Marshallese stick chart.

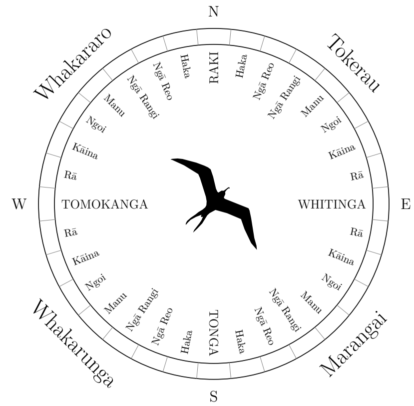
1.1.4 Elsewhere in the Pacific

In addition to the star compass, many cultures across the Pacific use a *wind compass*. Similar to the star compass, the wind compass is also a mental construct.

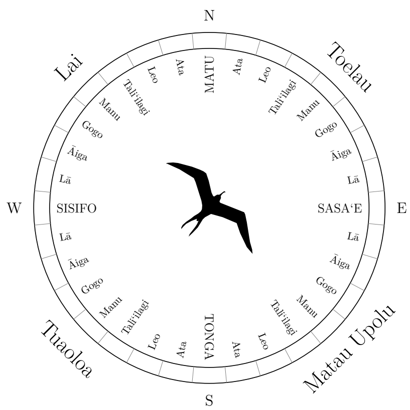
Other Pacific Island cultures have also adapted the modern Hawaiian Star Compass to their languages, as illustrated in Figure 1.1.9.



(a) Cook Islands.



(b) Māori.



(c) Sāmoa.

Figure 1.1.9 Examples of Star Compasses across the Pacific.

1.1.5 Exercises

1. Who developed the Hawaiian Star Compass?

Answer. Nainoa Thompson

2. The Hawaiian Star Compass was based on the Micronesian Star Compass, known as the *paafu*. Who shared the paafu with the Hawaiians?

Answer. Mau Piailug or Papa Mau

3. According to the Hawaiian Star Compass, what is the name for

(a) North

Answer. ʻĀkau

(b) East

Answer. Hikina

(c) South

Answer. Hema

(d) West

Answer. Komohana

4. What is the Hawaiian name for winds in the

(a) Northeast quadrant

Answer. Koʻolau

(b) Southeast quadrant

Answer. Malanai

(c) Southwest quadrant

Answer. Kona

(d) Northwest quadrant

Answer. Hoʻolua

Exercise Group. For each direction, identify the Hawaiian names of the corresponding house and quadrant in the Hawaiian Star Compass.

5. Northwest by North

Answer. Nālani Hoʻolua

6. East-northeast

Answer. ʻĀina Koʻolau

7. North-northwest

Answer. Nā Leo Hoʻolua

8. Southeast

Answer. Manu Malanai

9. South by West

Answer. Haka Kona

10. East by South

Answer. Lā Malanai

11. Southwest by West

Answer. Noio Kona

12. Northeast by North

Answer. Nālani Koʻolau

13. South-southeast

Answer. Nā Leo Malanai

14. West-southwest

Answer. ʻĀina Kona

Exercise Group. Identify the corresponding point on the 32-wind compass for each house on the Hawaiian Star Compass.

15. Nā Leo Kona

Answer. South-southwest

16. ʻĀina Malanai

Answer. East-southeast

17. ʻĀina Hoʻolua

Answer. West-northwest

18. Lā Koʻolau

Answer. East by North

19. Manu Koʻolau

Answer. Northeast

20. Haka Malanai

Answer. South by East

21. Nālani Kona

Answer. Southwest by South

22. Haka Hoʻolua

Answer. North by West

23. Noio Hoʻolua

Answer. Northwest by West

24. Noio Koʻolau

Answer. Northeast by East

25. The winter solstice in the southern hemisphere occurs around June 22. It is the time when the sun is at its lowest elevation in the sky, resulting in the shortest daylight of the year. During the winter solstice, the sun rises from its northernmost position, ʻĀina Koʻolau. In which house does the sun set during the winter solstice in the southern hemisphere?

Answer. ʻĀina Hoʻolua

26. The winter solstice in the northern hemisphere occurs around December 22 when the sun rises from its southernmost position, ʻĀina Malanai. In which house does the sun set during the winter solstice in the northern hemisphere?

Answer. ʻĀina Kona

27. Wind is coming from Nā Leo Kona. In which direction is the wind blowing?

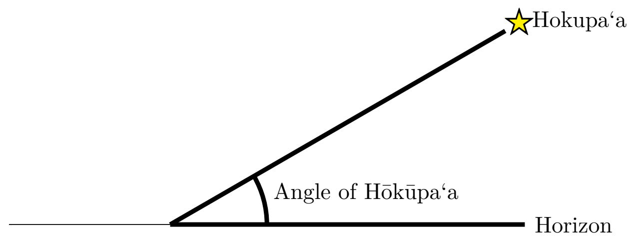
Answer. Nā Leo Koʻolau

28. Current is coming from Noio Hoʻolua. In which direction is the current heading?

Answer. Noio Malanai

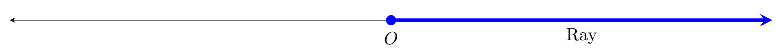
1.2 Angles and Their Measure

One method people use to identify their position is by looking at the *latitude*. These imaginary lines form circles around the Earth and run parallel to the Equator. The latitude of a place is defined as the angle between a line drawn from the center of the Earth to that point and the equatorial plane. For any point in the Northern Hemisphere, a navigator can measure their latitude by determining the angle that *Hōkūpa‘a* (also known as *Kūmau*, *Wuliwulifasmughet*, *Fuesemagut*, North Star, or Polaris) makes with the horizon.



During voyages, knowing the correct angles can make the difference between reaching our destination or missing it. Navigators carefully observe angles on the Hawaiian Star Compass to determine the entry and exit points of celestial bodies in the sky, as well as the direction of wind and current. In this section, we will explore the properties of angles and their measure.

Definition 1.2.1 A **ray** is a part of a line that begins at a point O and extends in one direction.



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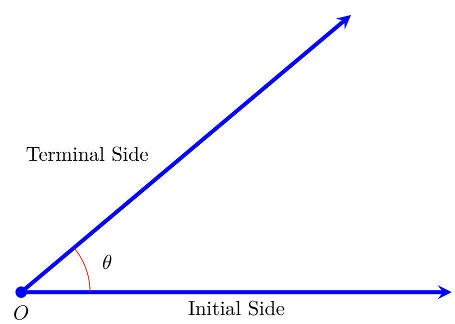
Definition 1.2.2 We can create an **angle**, θ, by rotating rays. First, we begin with two rays lying on top of each other and beginning at O. We let one ray be fixed and will rotate the second ray about the point O. The ray that is fixed is called the **initial side** and the ray that is rotated is called the **terminal side**.



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Remark 1.2.3 Angles are often measured using Greek letters. The commonly used Greek letters include θ, ϕ, α, β, and γ.

1.2.1 Degree

The **measure** of an angle is the amount of rotation from the initial side to the terminal side. One unit of measuring angles is the degree. One **degree**, denoted by $1°$, is $\frac{1}{360}$ of a complete circular revolution, so one full revolution is $360°$.

The Hawaiian Star Compass consists of 32 houses, each spanning 11.25° ($\frac{360°}{32}$). Assuming due East corresponds to 0° and the center of the House of Hikina points due East, the border between Hikina and Lā Koʻolau will be half the angle of the house, 5.625° ($\frac{11.25°}{2}$). The angles for the other boundaries on the Hawaiian Star Compass are shown in Figure 1.2.4.

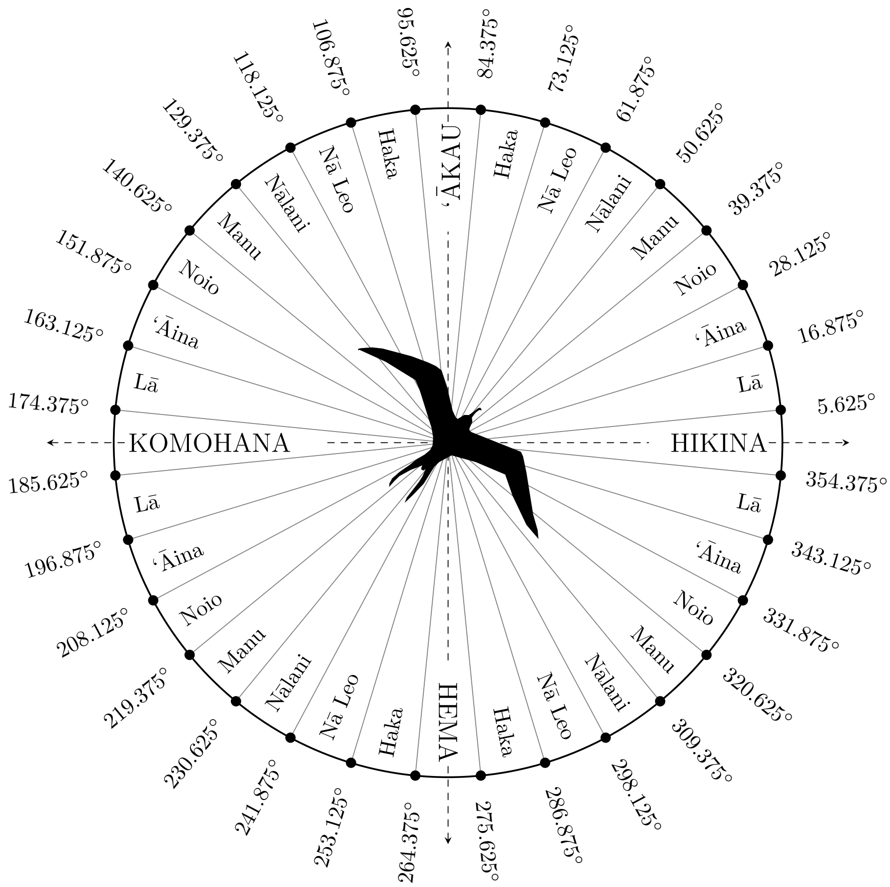


Figure 1.2.4 The Star Compass with the angles indicating the boundaries for each house.

Although decimals are commonly used to represent fractional parts of a degree, traditionally, degrees were represented in minutes and seconds. One **minute** or **arc minute**, denoted as $1'$, is equal to $\frac{1}{60}$ degrees, and one **second** or **arc second**, denoted as $1''$, is equal to $\frac{1}{60}$ minutes.

Remark 1.2.5 Conversion Between Degree, Minutes, and Seconds.

$$1° = 60' \qquad 1' = \left(\frac{1}{60}\right)^{\circ} \qquad 1'' = \left(\frac{1}{3600}\right)^{\circ}$$

$$1° = 3600'' \qquad 1' = 60'' \qquad 1'' = \left(\frac{1}{60}\right)'$$

Example 1.2.6 Convert an angle from decimal degrees to degrees/minutes/seconds. In the Star Compass (Figure 1.2.4), the angle between the houses Manu Hoʻolua (northwest) and Noio Hoʻolua (northwest by west) measures 140.625°. Represent this angle in degrees, minutes, and seconds.

Solution. First we will convert 0.625° to minutes using the conversion 1° =

$60'$,

$$0.625° = 0.625° \cdot \frac{60'}{1°} = 37.5'.$$

Since $1' = 60''$, we can convert $0.5'$ to seconds: $0.5' = 0.5' \cdot \dfrac{60''}{1'} = 30''$.

So $140.625° = 140°37'30''$. □

Example 1.2.7 Convert an angle from degrees/minutes/seconds to decimal degrees. Convert $263°24'45''$ to decimal degrees.

Solution. We will first convert $24'$ and $45''$ to degrees.

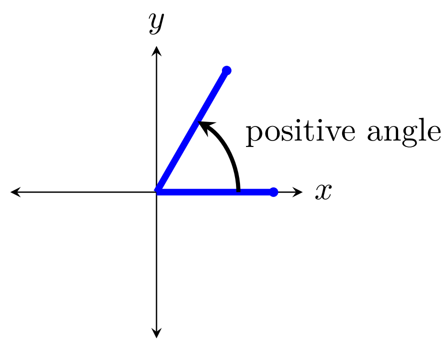
$$24' = 24 \cdot 1' = 24 \cdot \left(\frac{1}{60}\right)^{\circ} = 0.4°$$

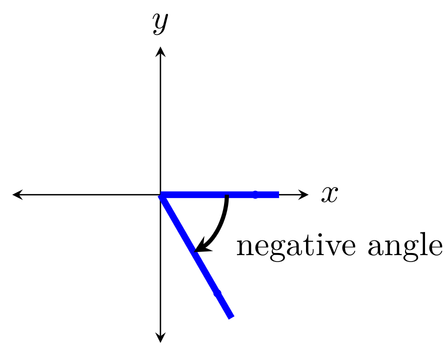
and

$$45'' = 45 \cdot 1'' = 45 \cdot \left(\frac{1}{3600}\right)^{\circ} = 0.0125°.$$

So $263°24'45'' = 263° + 24' + 45'' = 263° + 0.4° + 0.0125 = 263.4125°$. □

Definition 1.2.8 If an angle is drawn on the xy-plane, and the vertex is at the origin, and the initial side is on the positive x-axis, then that angle is said to be in **standard position**. If the angle is measured in a counterclockwise rotation, the angle is said to be a **positive angle**, and if the angle is measured in a clockwise rotation, the angle is said to be a **negative angle**.





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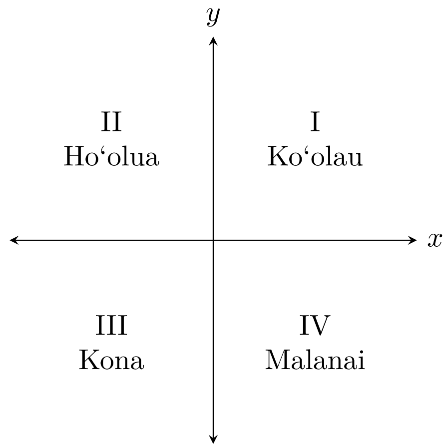
Definition 1.2.9 When an angle is in standard position, the terminal side will either lie in a quadrant or it will lie on the x-axis or y-axis. An angle is called a **quadrantal angle** if the terminal side lies on x-axis or y-axis. The two axes divide the plane into four **quadrants**. In the Cartesian plane, the four quadrants are Quadrant I, II, III, and IV. The corresponding quadrants of Star Compass are Koʻolau (NE), Hoʻolua (NW), Kona (SE), and Malanai (SW).



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Definition 1.2.10 Coterminal angles are angles in standard position that have the same initial side and the same terminal side. Any angle has infinitely many coterminal angles because each time we add or subtract 360° from it, the resulting angle has the same terminal side. ◊

Example 1.2.11 Coterminal angles. 90° and 450° are coterminal angles since $450° - 360° = 90°$. □

To determine the quadrant in which an angle lies, we add or subtract one revolution (360°) until we obtain a coterminal angle between 0° and 360°. The quadrant where the terminal side lies is the quadrant of the angle. Quadrantal angles do not lie in any quadrant.

Example 1.2.12 Determine the corresponding house and quadrant in the Star Compass. Determine the quadrant in which each angle lies and name the corresponding house and quadrant in the Star Compass (Figure 1.2.4).

(a) 140°

Solution. Since $90° < 140° < 180°$, 140° lies in Quadrant II, or Manu Hoʻolua.

(b) −770°

Solution. Since $-770° < 0°$, we first add 3×360 to −770° to obtain an angle 0° and 360°,

$$-770° + 3 \times 360° = 310°.$$

So 310° and −770° are coterminal. Since $270° < 310° < 360°$, 310° lies in Quadrant IV, or Manu Malanai.

(c) 923°

Solution. Since $923° > 360°$ we begin by subtracting $2 \times 360°$

$$923° - 2 \times 360° = 203°.$$

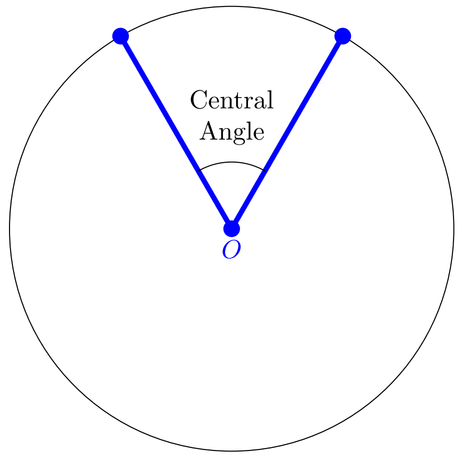
So 203° and 923° are coterminal. Since $180° < 203° < 270°$, 923° lies in Quadrant III or ʻĀina Kona.

□

Example 1.2.13 Determine the corresponding quadrant given its location in the Star Compass. What is the corresponding quadrant for Nālani Kona?

Solution. Locating Nālani Kona in the Star Compass, we see it is in Quadrant III. □

Definition 1.2.14 A **central angle** is an angle formed at the center of a circle by two radii.



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Remark 1.2.15 Heading and Azimuth. In navigation, the direction a wa'a is pointed towards is referred to as the **heading**. Unlike in trigonometry, where it is conventional to define an angle in standard position, with 0° lying along the positive x-axis, in navigation, North corresponds to a heading of 0° and positive angles are measured in a clockwise rotation (see Figure 1.2.16).

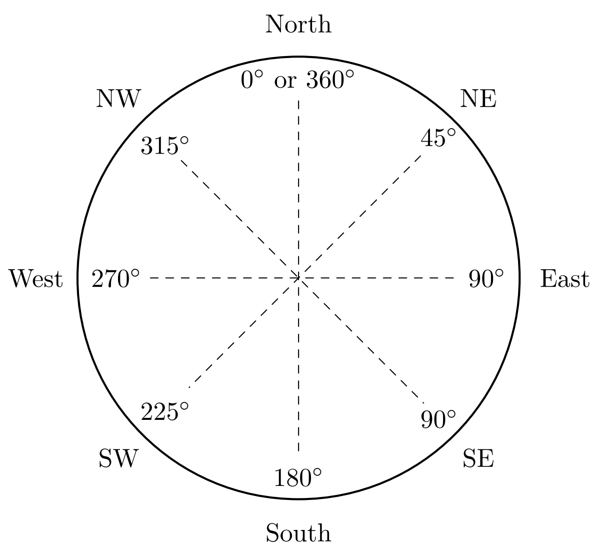


Figure 1.2.16 The cardinal directions for headings are as follows: 0° (or 360°) points north, 90° points east, 180° points south, and 270° points west.

The Star Compass can now be presented in terms of heading angles, as demonstrated in Figure 1.2.17.

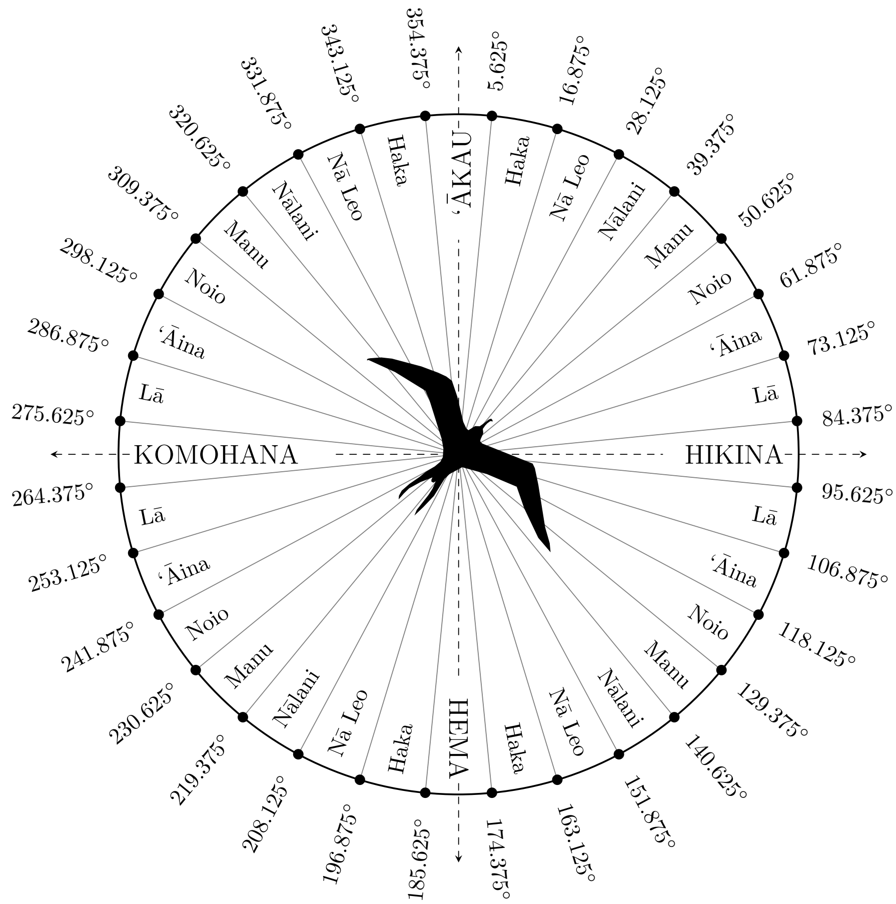


Figure 1.2.17 The Star Compass is presented in terms of heading, with the angles indicating the boundaries for each house.

In astronomy and navigation, the **azimuth** and **altitude** are used to determine the position of celestial bodies relative to the observer's location. The azimuth refers to the angular measurement of the direction of a celestial body, typically measured from north and increasing clockwise along the horizon. The altitude of a celestial body is its angle above the horizon, measured in degrees. The **zenith** is the point directly overhead an observer, and a celestial body located at the zenith has an altitude of 90°. See Figure 1.2.18.

In navigation, "heading" typically refers to the direction in which an object, such as a canoe or wind, is pointed. In contrast, "azimuth" pertains to the angular measurement of celestial bodies relative to the horizon. Both "heading" and "azimuth" measure angles in degrees, starting from north and progressing clockwise. Unless specified otherwise, this book uses Figure 1.2.4 for the angles of the Star Compass; for heading or azimuth angles, refer to Figure 1.2.17.

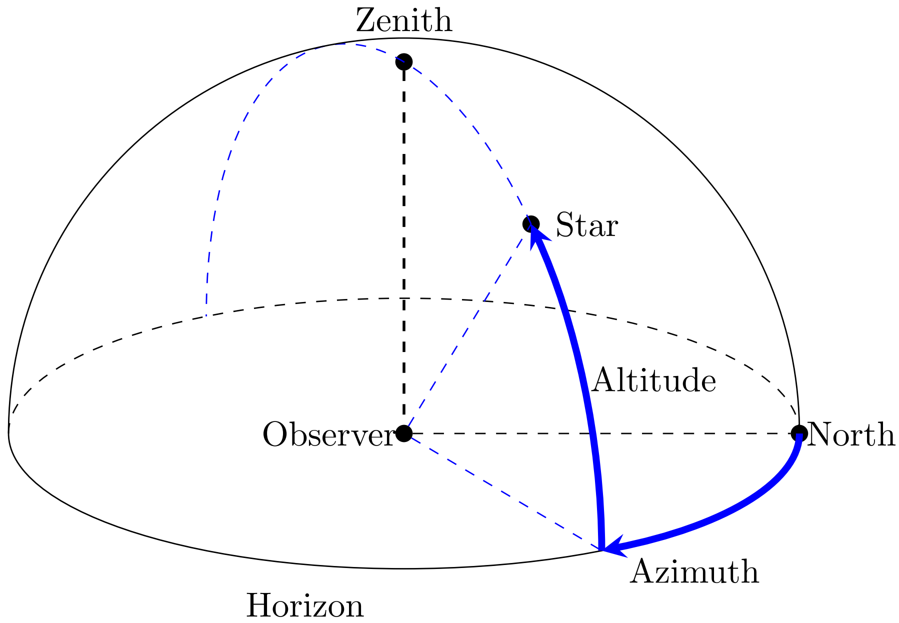


Figure 1.2.18 The azimuth is the angle measured along the horizon from north to the star's projection, and the altitude is the angle from the horizon up to the star. Together, these angles determine the star's location relative to the observer.

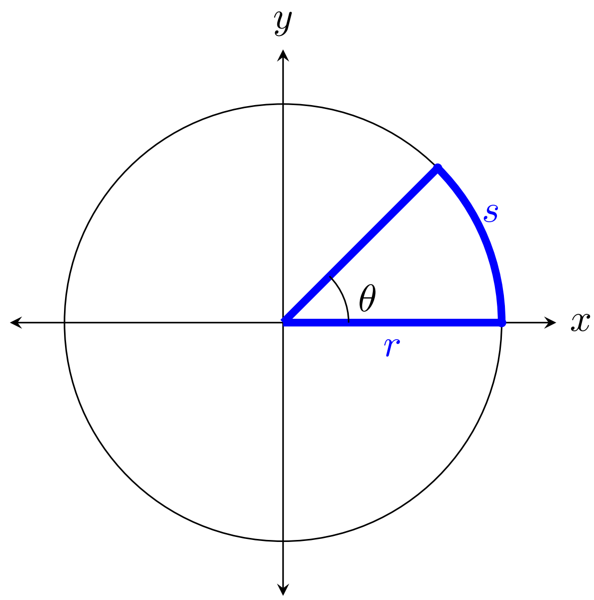
1.2.2 Radian

Another way to measure an angle is with *radians*, which measure the arc of a circle that is formed from an angle.

Definition 1.2.19 Definition of a Radian. The **radian measure** of a central angle in a circle is the ratio of the length of the arc on the circle subtended by the angle to the radius. If r is the radius of the circle, θ is the angle, and s is the arc length, then we have the following

$$\theta = \frac{s}{r}.$$

A radian is abbreviated by **rad**.



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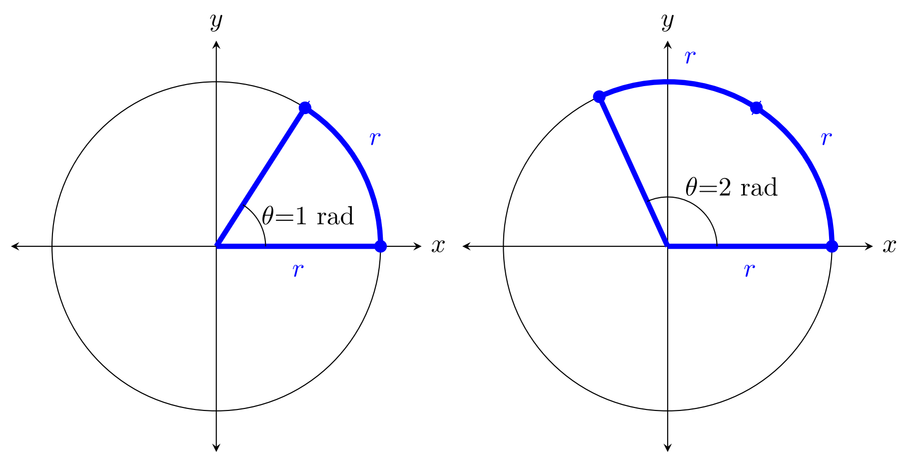
The measure of a central angle obtained when the length of the arc is also equal to the radius, r, is called *one radian* (1 rad). Similarly, if $\theta = 2$ rad, then the arc length equals $2r$.



The circumference of a circle is $C = 2\pi r$. This means that the circumference is $2\pi \approx 6.28$ times the radius. Consequently, if we were to use a string of length equal to the radius, we would need six such strings plus a fractional piece of the string, as shown in Figure 1.2.20.

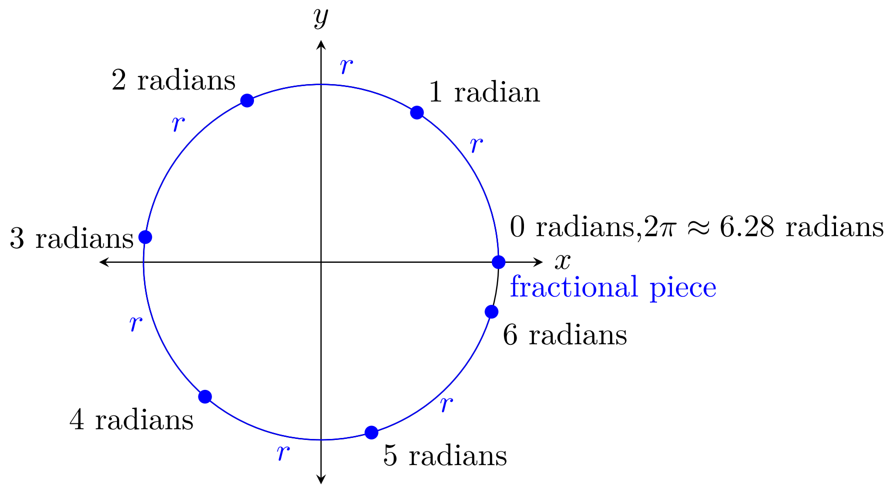


Figure 1.2.20 One rotation of the unit circle is $2\pi \approx 6.28$ radians.

Remark 1.2.21 Relationships Between Degrees and Radians. If a circle with radius 1 is drawn, it has $360°$, and the full arc length is the circumference, which is 2π. Therefore,the relationship between degrees and radians is:

$$360° = 2\pi \text{ radian, or } 180° = \pi \text{ radian}$$

$$1 \text{ radian} = \frac{180°}{\pi},$$

$$1° = \frac{\pi}{180} \text{ radian.}$$

Remark 1.2.22 Converting Between Degrees and Radians.

1. To convert degrees to radians, multiply by $\dfrac{2\pi \text{ radians}}{360°}$ or $\dfrac{\pi \text{ radian}}{180°}$.
2. To convert radians to degrees, multiply by $\dfrac{360°}{2\pi \text{ radians}}$ or $\dfrac{180°}{\pi \text{ radian}}$.

Example 1.2.23 Express $45°$ in radians.

Solution. $45° = 45°\left(\dfrac{2\pi \text{ radians}}{360°}\right) = \dfrac{\pi}{4}$ radians □

Example 1.2.24 Express $\dfrac{5\pi}{6}$ in degrees.

Solution. $\dfrac{5\pi}{6}\text{rad} = \dfrac{5\pi}{6}\text{rad}\left(\dfrac{360°}{2\pi \text{ rad}}\right) = 150°$ □

Using this method, we obtain Table 1.2.25 of common angles used in trigonometry and the corresponding radian and degree measures.

Table 1.2.25 Commonly Used Angles in Trigonometry: Degrees and Radians.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
Radians	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degrees	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$

1.2.3 Arc Length

Recall that the definition of a radian is the ratio of the arc length to the radius of a circle, $\theta = \frac{s}{r}$. By rearranging this formula, we can obtain a formula for the arc length of a circle.

Theorem 1.2.26 *In a circle of radius r, the* arc length*, s, subtended by a central angle (in radians), θ, is*

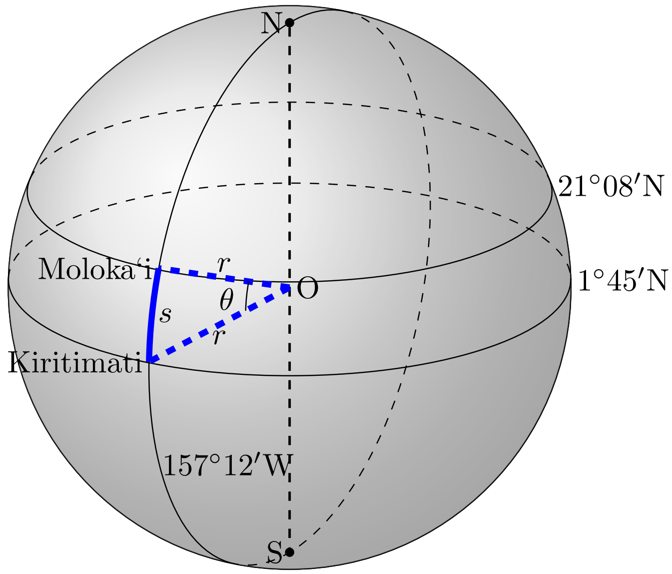
$$s = r\theta.$$

If θ is given in degrees, then $s = 2\pi r \cdot \left(\frac{\theta}{360°}\right)$.

Example 1.2.27 Find the length of an arc of a circle with radius 10 cm subtended by an angle of 2 radians.

Solution. Using the arc length formula we get $s = 10\text{cm} \cdot 2\text{rad} = 20\text{cm}$. □

Example 1.2.28 Kiritimati, also known as Christmas Island, is an atoll in the Republic of Kiribati. Kiritimati's location west of the International Date Line makes it one of the first places in the world to welcome the New Year, while Hawaiʻi is one of the last places. Although Kiritimati and Molokaʻi share the same longitude at $157°12'$ west (meaning Molokaʻi is directly north of Kiritimati), both islands are 24 hours apart. For example, if the time on Molokaʻi is 3:00 pm on Thursday, then at that same moment it is 3:00 pm on Friday in Kiritimati. Find the distance between Kiritimati ($1°45'$ north latitude) and Molokaʻi ($21°08'$ north latitude). Assume the radius of Earth is 3,960 miles and that the central angle between the two islands is the difference in their laititudes.



Solution. The measure of the central angle between the two islands is

$$\begin{aligned}\theta &= 21°08' - 1°45' \\ &= 19°23' \\ &= 19° + \frac{23}{60}° \\ &\approx 19.3833°.\end{aligned}$$

To find the distance, we use Theorem 1.2.26 to find the arc length:

$$s = 2\pi r \cdot \left(\frac{\theta}{360°}\right) \approx 2\pi \cdot (3,960 \text{ miles})\frac{19.3833°}{360°} \approx 1,340 \text{ miles}.$$

So the distance between Kiritimati and Moloka'i is approximately 1,340 miles. □

1.2.4 Area of a Sector of a Circle

Definition 1.2.29 Area of a Sector. The **area of the sector** of a circle of radius r formed by a central angle of θ is

$$\begin{aligned}A &= \frac{\theta}{360°} \cdot \pi \cdot r^2, \text{ when } \theta \text{ is in degrees} \\ A &= \frac{1}{2}r^2\theta, \text{ when } \theta \text{ is in radians.}\end{aligned}$$

◊

Notice the ratio $\frac{\theta}{360°}$ is the proportion of the angle θ (in degrees) to one complete circle. Additionally, the circumference of a circle is given by $2\pi r$. Therefore, the arc length is simply the proportion of the central angle to the whole circle multiplied by the circumference of the circle.

$$s = \text{arc length} = (\text{proportion of circle}) \cdot (\text{circumference}) = \left(\frac{\theta}{360°}\right) \cdot (2\pi r)$$

Similarly, the area of a circle is given by πr^2. So the area of a sector is the proportion of the central angle to the whole circle multiplied by the area of the circle.

$$A = \text{area of sector} = (\text{proportion of circle}) \cdot (\text{area of circle}) = \left(\frac{\theta}{360^\circ}\right) \cdot \left(\pi r^2\right)$$

Theorem 1.2.30 *Given a circle of radius r and a central angle θ, the arc length and area of the sector formed by θ can be expressed as the proportion of the angle to the full circle multiplied by the circumference and area of the circle, respectively.*

$$s = \textit{(proportion of circle)} \cdot \textit{(circumference)} = \left(\frac{\theta}{360^\circ}\right) \cdot (2\pi r)$$

and

$$A = \textit{(proportion of circle)} \cdot \textit{(area of circle)} = \left(\frac{\theta}{360^\circ}\right) \cdot \left(\pi r^2\right).$$

Example 1.2.31 When sailing, the waʻa Hōkūleʻa cannot make headway by sailing directly into the wind. She can only sail beyond 67° in either direction from the wind (Figure 1.2.32). If Hōkūleʻa sails 50 miles, what is the area of the sector that cannot be sailed? Round your answer to the nearest square mile.

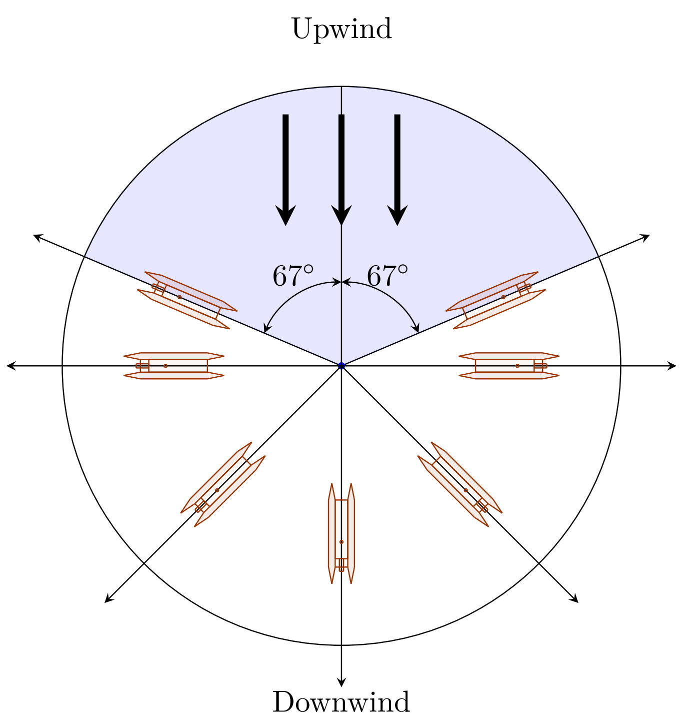


Figure 1.2.32 Hōkūleʻa cannot sail within 67° into the direction of the wind.

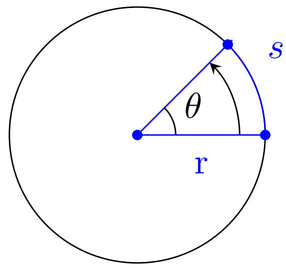
Solution. The angle is $\theta = 2 \cdot 67^\circ = 134^\circ$ and the radius is $r = 50$ miles. So the area is given by

$$A = \frac{\theta}{360^\circ}\pi \cdot r^2 = \frac{134^\circ}{360^\circ}\pi \cdot 50^2 \approx 2,923 \text{ square miles.}$$

□

1.2.5 Angular Velocity and Linear Speed

Consider an object moving along a circle as shown below. There are two ways to describe the circular motion of this object: *linear speed* which measures the distance traveled; and *angular speed* which measures the rate at which the central angle changes.



Definition 1.2.33 Linear Speed. Suppose an object moves along a circle with radius r, and let θ (measured in radians) be the angle traversed in time t. Let s be the distance the object traveled in time t. Then the **linear speed**, v, of the object is given by

$$v = \frac{s}{t}.$$

$\Diamond$

Definition 1.2.34 Angular Speed. Suppose an object moves along a circle. Let θ (measured in radians) be the angle traversed by the object in time t. The **angular speed**, ω, of the object is given by

$$\omega = \frac{\theta}{t}.$$

$\Diamond$

Notice that we can rearrange the angular speed to get $\theta = \omega t$. Since s is an arc length, we have $s = r\theta$, and thus we can write the linear speed as

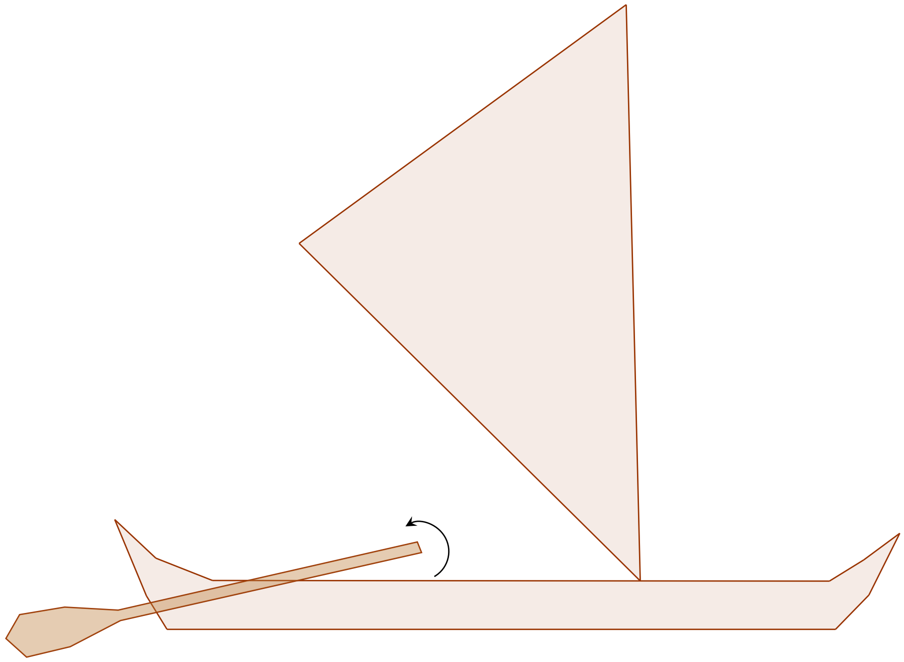
$$v = \frac{s}{t} = \frac{r\theta}{t} = \frac{r\omega t}{t} = r\omega.$$

Definition 1.2.35 Linear Speed. Suppose an object moves along a circle with radius r and an angular speed ω (measured in radians per unit time). Then the **linear speed**, v, of the object is given by

$$v = r\omega.$$

$\Diamond$

Example 1.2.36 Une.



One method a wa‘a uses to change direction is with the *hoe uli*, or the steering paddle. When a sharp turn is needed for maneuvers such as tacking, the steersperson will turn the handle of the hoe uli in a circular motion, as a lever to scoop the paddle in the water and change the heading of a vessel. This move, called *une* (pronounced "oo-neh," though it is often mispronounced as "oo-nee"), literally translates to "lever."

Standalone

Figure 1.2.37 A wa‘a (canoe) can change directions by rotating the hoe uli (steering sweep) in a process known as une. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to watch the video online.

If the steerperson is performing an une at a rate of 25 rotations per minute and the radius of the circular movement is 2 feet, calculate:

(a) The angular speed measured in radians per minute.

Solution. We are given that the angular speed is $\omega = 25$ revolutions per minute. To convert our angular speed to radians per minute, we use the fact that one revolution is 2π radians to get

$$\omega = 25\frac{\text{revolution}}{\text{minute}} = 25\frac{\text{revolution}}{\text{minute}} \cdot 2\pi\frac{\text{radians}}{\text{revolution}} = 50\pi\frac{\text{radians}}{\text{minute}}.$$

Thus the hoe uli is moving at an angular speed of 50π radians per minute.

(b) The linear speed of the hoe uli in miles per hour (round your answer to two decimal places).

Solution. Since the radius is $r = 2$ ft and the angular speed is 50π radians per minute, we can use Definition 1.2.35 to calculate the linear speed

$$v = r\omega = 2\text{ft} \cdot 50\pi\frac{\text{radians}}{\text{min}} \cdot \frac{\text{mile}}{5280\text{ft}} \cdot \frac{60\text{min}}{\text{hr}} \approx 3.57\frac{\text{miles}}{\text{hour}}.$$

Thus the steersperson is moving the hoe uli at a linear speed of 3.57 mph.

□

1.2.6 Exercises

Exercise Group. Given an angle, θ, identify the house and quadrant on the Hawaiian Star Compass.

1. $\theta = 336°$

Answer. ʻĀina Malanai

2. $\theta = 240°$

Answer. Nālani Kona

3. $\theta = 35°$

Answer. Noio Koʻolau

4. $\theta = 221°$

Answer. Manu Kona

5. $\theta = 108°$

Answer. Nā Leo Hoʻolua

6. $\theta = 190°$

Answer. Lā Kona

7. $\theta = 323°$

Answer. Noio Malanai

8. $\theta = 158°$

Answer. ʻĀina Hoʻolua

9. $\theta = 172°$

Answer. Lā Hoʻolua

Exercise Group. Convert the given angle θ to a decimal in degrees rounded to two decimal places.

10. $\theta = 20°50'30''$

Answer. 20.84°

11. $\theta = 80°25'16''$

Answer. 80.42°

12. $\theta = 7°22'38''$

Answer. 7.38°

13. $\theta = 330°14'12''$

Answer. 330.24°

14. $\theta = 168°22'26''$

Answer. 168.37°

15. $\theta = 49°27'12''$

Answer. 49.45°

16. $\theta = 327°38'58''$

Answer. 327.65°

17. $\theta = 281°48'50''$

Answer. 281.81°

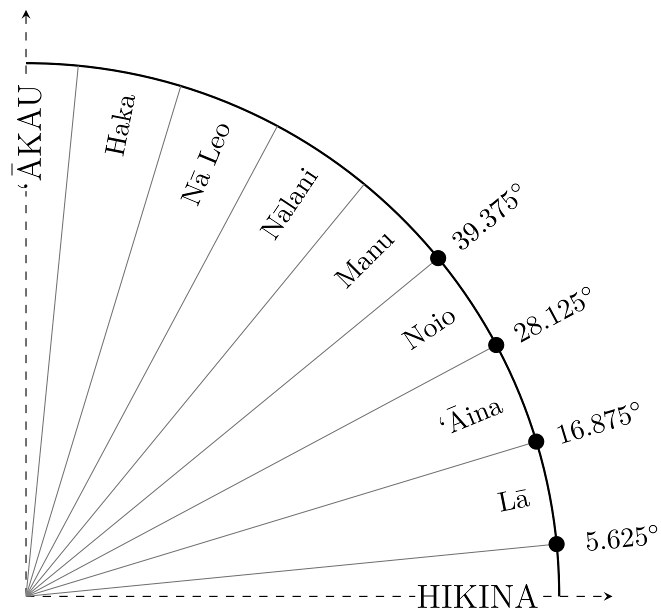
18. $\theta = 134°53'47''$

Answer. 134.90°

19. Sirius, the brightest star in the night sky, has been known by various names in different cultures and languages around the world. In Tahiti, it is called Taurere and is considered a zenith star as it passes directly overhead. Taurere has a **declination** of $-16°42'58''$, representing its angular distance south of the celestial equator. Express Taurere's declination as a decimal rounded to two decimal places.

Answer. 16.72°

Exercise Group. Recall the Star Compass with the boundaries for each House. Write the angles for the boundaries between the following houses in the Koʻolau quadrant in terms of degrees, minutes, and seconds.



20. Between Hikina and Lā (5.625°)

Answer. $5°37'30''$

21. Between Lā and ‘Āina (16.875°)

Answer. $16°52'30''$

22. Between ‘Āina and Noio (28.175°)

Answer. $28°10'30''$

23. Between Noio and Manu (39.375°)

Answer. $39°22'30''$

Exercise Group. Convert the given angle θ to degrees/minutes/seconds rounded to the nearest second and identify the house and quadrant on the Hawaiian Star Compass.

24. $\theta = 258.39°$

Answer. $\theta = 258°23'24''$ Haka Kona

25. $\theta = 212.43°$

Answer. $\theta = 212°25'33''$ Noio Kona

26. $\theta = 244.97°$

Answer. $\theta = 244°58'48''$ Nā Leo Kona

27. $\theta = 93.95°$

Answer. $\theta = 93°57'0''$ ‘Ākau

28. $\theta = 135.625°$

Answer. $\theta = 135°37'30''$ Manu Ho‘olua

29. $\theta = 162.52°$

Answer. $\theta = 162°31'12''$ ‘Āina Ho‘olua

30. $\theta = 328.21°$

Answer. $\theta = 328°12'36''$ Noio Malanai

31. $\theta = 48.12°$

Answer. $\theta = 48°7'12''$ Manu Ko‘olau

32. $\theta = 241.27°$

Answer. $\theta = 241°16'12''$ Nā Lani Kona

Exercise Group. Convert the given angle θ to radians, and identify the house and quadrant on the Hawaiian Star Compass. Express your answers in terms of π.

33. $\theta = 102^\circ$

Answer. $\theta = \frac{17\pi}{30}$ Haka Hoʻolua

34. $\theta = 96^\circ$

Answer. $\theta = \frac{8\pi}{15}$ ʻĀkau

35. $\theta = 190^\circ$

Answer. $\theta = \frac{19\pi}{18}$ Lā Kona

36. $\theta = 122^\circ$

Answer. $\theta = \frac{61\pi}{90}$ Nālani Hoʻolua

37. $\theta = 32^\circ$

Answer. $\theta = \frac{8\pi}{45}$ Noio Koʻolau

38. $\theta = 236^\circ$

Answer. $\theta = \frac{59\pi}{45}$ Nālani Kona

39. $\theta = 82^\circ$

Answer. $\theta = \frac{41\pi}{90}$ Haka Koʻolau

40. $\theta = 228^\circ$

Answer. $\theta = \frac{19\pi}{15}$ Manu Kona

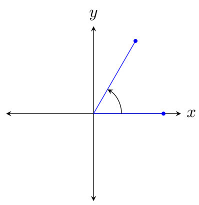
41. $\theta = 332^\circ$

Answer. $\theta = \frac{83\pi}{45}$ ʻĀina Malanai

Exercise Group. Draw the angle in standard position.

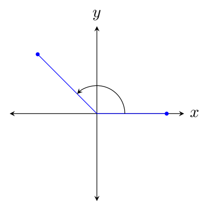
42. 60°

Answer.



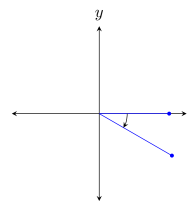
43. 135°

Answer.



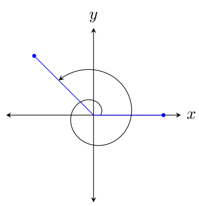
44. -30°

Answer.



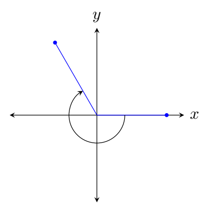
45. 495°

Answer.



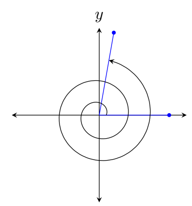
46. -240°

Answer.



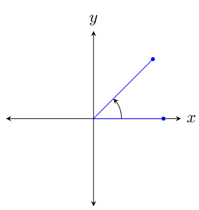
47. 800°

Answer.



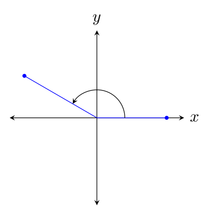
48. $\frac{\pi}{4}$

Answer.



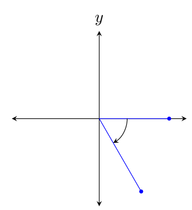
49. $\frac{5\pi}{6}$

Answer.



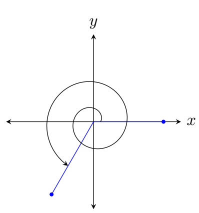
50. $-\frac{\pi}{3}$

Answer.



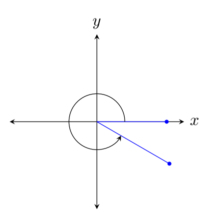
51. $\frac{10\pi}{3}$

Answer.



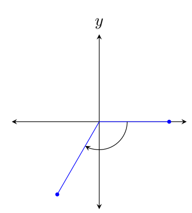
52. $\frac{11\pi}{6}$

Answer.



53. $-\frac{2\pi}{3}$

Answer.



Exercise Group. Convert the given angle from degrees to radians. Round your answer to two decimal places.

54. $27°$

Answer. 0.47

55. $63°$

Answer. 1.10

56. $-39°$

Answer. -0.68

57. $200°$

Answer. 3.49

58. $415°$

Answer. 7.24

59. $105°$

Answer. 1.83

Exercise Group. Convert the given angle from radians to degrees. Round your answer to two decimal places.

60. $\frac{3\pi}{5}$

Answer. $108°$

61. $\frac{8\pi}{3}$

Answer. $480°$

62. $-\frac{4\pi}{7}$

Answer. $-102.86°$

63. $\frac{15\pi}{4}$

Answer. $675°$

64. -1.5

Answer. $-85.94°$

65. 3

Answer. $171.89°$

Exercise Group. Determine whether the two given angles in standard position are coterminal.

66. $50°$, $410°$

Answer. Yes

67. $-60°$, $330°$

Answer. No

68. $30°$, $1110°$

Answer. Yes

69. $-210°$, $150°$

Answer. Yes

70. $-807°$, $93°$

Answer. No

71. $-757°$, $683°$

Answer. Yes

Exercise Group. Find an angle between $0°$ and $360°$ that is coterminal with the given angle.

72. $405°$

Answer. $45°$

73. $600°$

Answer. $240°$

74. $1035°$

Answer. $315°$

75. $-300°$

Answer. $60°$

76. $381°$

Answer. $21°$

77. $-754°$

Answer. $326°$

Exercise Group. Given a circle with radius r, calculate (a) the length of the arc subtended by a central angle θ; and (b) the area of a sector with central angle θ. Round your answer to four decimal places.

78. $r = 10$ in, $\theta = 45°$

Answer.

(a) 7.8540 in;

(b) 39.2699 in^2

79. $r = 5$ m, $\theta = 120°$

Answer.

(a) 10.4720 m;

(b) 26.1799 m^2

80. $r = 3$ mi, $\theta = \frac{\pi}{3}$ radians

Answer.

(a) 3.1416 mi;

(b) 4.7124 mi^2

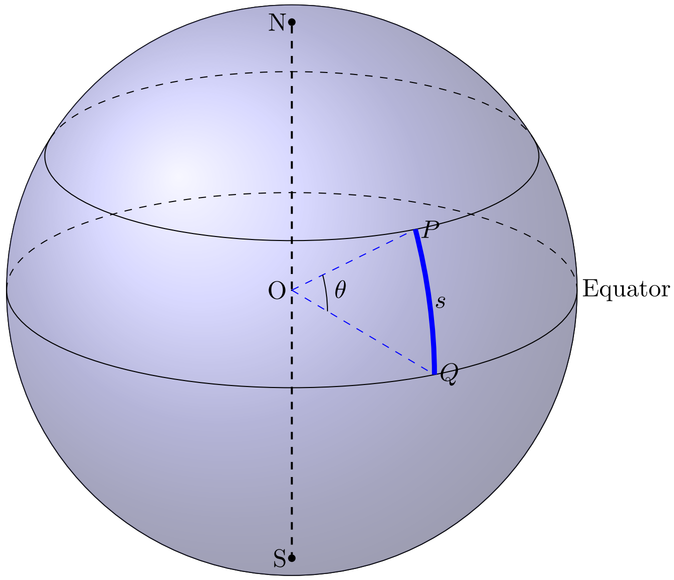
81. $r = 8$ cm, $\theta = 2$ radians

Answer.

(a) 16 cm;

(b) 64 cm^2

Exercise Group. At the start of this section, we learned that the **latitude** of a place is the angle between a line drawn from the center of the earth to that point and the equatorial plane. If the radius of the Earth is 3,959 miles, calculate the arc length, s, along the surface of the earth for each value of θ.



82. $\theta = 1^\circ$ of latitude (in miles, rounded to 2 decimals).

Answer. 69.10 miles

83. $\theta = 1'$ of latitude (in miles, rounded to 2 decimals). A *nautical mile*, frequently used in navigation, is slightly longer than a mile on land. One nautical mile was historically defined to be the arc length corresponding to one minute of latitude. Compare your result with the accepted value of one nautical mile.

Answer. 1.15 miles

84. $\theta = 1''$ of latitude (in feet, rounded to the nearest integer). Note that 1 mile is equivalent to 5,280 feet.

Answer. 101 feet

85. The *oeoe*, or Hawaiian bullroarer, is made by drilling holes into a kamani seed or coconut shell, then threading a long string through the holes to secure it. When the oeoe is swung by the string, a whistling sound is produced, similar to the sound of the wind on the top of mountains. If a girl is swinging an oeoe at the end of 3 foot long rope at a rate of 180 revolutions per minute, calculate:

(a) The angular speed measured in radians per minute.

Answer. 360π radians/minute

(b) The linear speed of the shell in miles per hour (round to two decimal places).

Answer. 38.56 miles/hour

86. The Earth completes one revolution around the Sun approximately every 365.25 days. We will assume the orbit is a circle, and that the Earth is 92.9 million miles from the Sun.

(a) How far does the Earth travel in one day, expressed as millions of miles?

Hint. First determine the angle or proportion of a revolution that Earth travels in one day, then calculate the arc length of Earth's orbit.

Answer. 1.6 million miles

(b) How for does the Earth travel in 30 days, expressed as millions of miles?

Answer. 47.9 million miles

(c) How far does the Earth travel in one revolution around the sun, expressed as millions of miles?

Answer. 583.7 million miles

(d) What is the linear speed of Earth as it orbits the Sun? Express your answer in miles per hour.

Answer. $66,588$ miles per hour

87. As the moon orbits the Earth, different parts of its surface become illuminated by the Sun, which we call moon phases. The moon completes one revolution about the Earth in approximately 27.3 days. If we assume its orbit is circular and the moon is 239,000 miles from Earth, calculate the linear speed of the moon, expressed as miles per hour.

Answer. 2,292 miles per hour

88. At 17.7° S latitude, the city of Nadi, Fiji is $6,071$ km from the Earth's axis of rotation. In 24 hours, Nadi will have traveled one revolution around Earth or $2\pi \cdot (6,071)$ km. The city of Port Vila, Vanuatu lies 967 km directly to the west of Nadi, Fiji. As the Earth rotates, how many minutes sooner will the people in Nadi see the Sun rise than the people in Port Vila, rounded to one decimal?

Hint. The proportion of distance between the two cities to the distance traveled in one rotation is the same as the proportion of the time it takes to see the sun between the two cities to time it takes to complete one revolution.

Answer. 36.5 minutes

89. In Example 1.2.31, we learned that wa‘a cannot sail directly into the wind. For each of the following canoes and distance traveled, determine the area of the sector that cannot be sailed? Recall that 1 house $= 11.25°$. Round your answer to the nearest square mile.

(a) Makali‘i sails for 25 miles and cannot sail within 4 houses from the direction of the wind.

Answer. 491 square miles

(b) Alingano Maisu sails for 15 miles and cannot sail within 3 houses from the direction of the wind.

Answer. 133 square miles

90. Navigating by the Sun: Using Solar Declination and Rising Sun to Orient on a Canoe. The position of the rising or setting sun changes throughout the year. **Solar declination** (denoted as δ) is the angle between the direction where the Sun rises (or sets) and due east (or due west) on the horizon. It represents how far north or south the Sun is from the celestial equator, projected onto the Earth's equatorial plane. Solar declinations to the north are positive, while those to the south are negative. At the Equinoxes (around March 20th and September 22nd), the solar declination is $0°$ ($\delta = 0°$), as the Sun is directly above the equator. During the December solstice, around December 22, the Sun rises from its most southern position, 23.5 degrees south of due east ($\delta = -23.5°$), and during the June solstice, around June 22, the Sun rises from its most northern position, 23.5 degrees north of due east ($\delta = 23.5°$).

A navigator can use their knowledge of the rising sun to help orient themselves. For example, on May 22, the solar declination is $\delta = 20°16'$. If the navigator identifies where the Sun rises on the horizon during the Equinox, she can measure $20°16'$ south of that point to identify East and then orient herself accordingly.

(a) What is the azimuth of the Sun on May 22? Remember, the azimuth is the angular measurement from North, measuredclockwise.

Answer. $69°44'$

(b) What house is the sun rising in?

Answer. ʻĀina Koʻolau

(c) What house does the sun set in?

Hint. Celestial bodies rise and set in the same house but different quadrants.

Answer. ʻĀina Hoʻolua

(d) If the canoe is sailing with the rising sun on the port side (the left side of the canoe when facing forward), and the navigator measures the angle between the direction of the canoe and the sun as 90°, what is the heading of the canoe? Recall that heading is the angle measured clockwise from North.

Answer. $159°44'$

(e) What house is the canoe sailing in?

Answer. Nā Leo Malanai

91. Swells are one of the most consistent navigational tools used to keep on a course because they can remain constant over time. On 27 May 2023, while sailing on the vaka Paikea from Rarotonga to Apia, you finished your watch (your assigned shift) and are ready to take a nap. Before you lie down, you take note that the canoe has a heading of 310° and the swells are coming from the southwest (Manu Kona) and hitting the canoe at 5° above directly left of the canoe.

(a) What is the heading of the swell?

Answer. 225°

(b) When you wake, you noticed the swells are now hitting the canoe from 25° to the left of your heading. You are aware that the swells could not have changed this fast and conclude that while you were asleep, the canoe changed its heading. Assuming the swell was constant, determine your new heading.

Answer. 250°

92. After spending six weeks in Samoa, vaka Paikea is making his way back from Apia to Rarotonga. On July 14, 2023, the canoe is sailing with a heading of 50°, and the wind is coming to the canoe from 60° to the right of the heading. What is the heading and house from which the wind is coming?

Answer. 110°; ‘Āina Malanai

1.3 Unit Circle

In this section, we will introduce the trigonometric functions using the Unit Circle.

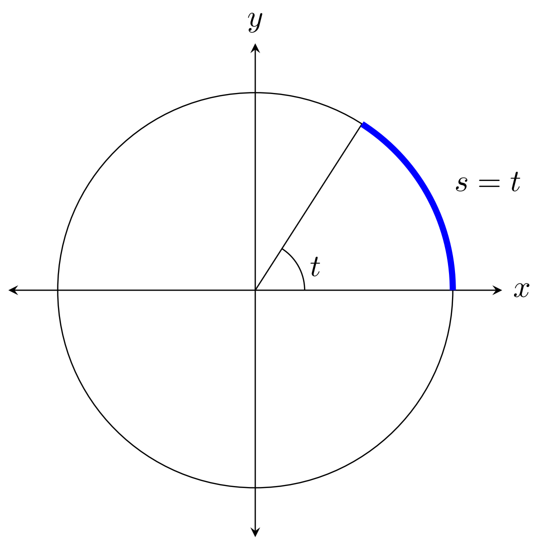
1.3.1 Unit Circle

Definition 1.3.1 Unit Circle. The **unit circle** is a circle whose radius is 1 and whose center is at the origin of a coordinate plane (or xy-plane). The equation for the unit circle is

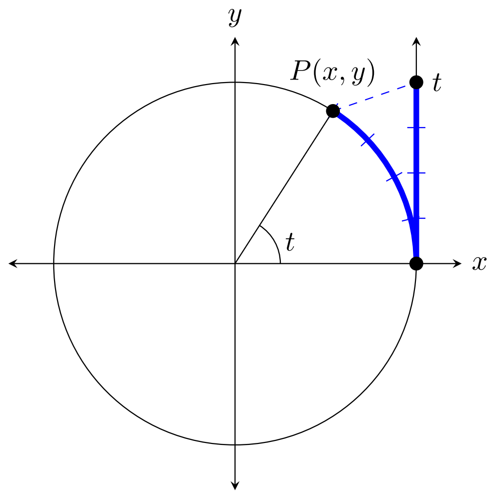
$$x^2 + y^2 = 1.$$

$\Diamond$

Let t be a real number. Recall from Definition 1.2.19 that the radian measure of a central angle, t, is defined as the ratio of the arc length s to the radius r. In other words, $t = \frac{s}{r}$. In the unit circle, the radius is $r = 1$, and the angle in radians is equal to the arc length, $t = s$. We will let t be in radians. The circumference of the unit circle is $2\pi r = 2\pi \cdot 1 = 2\pi$.



If $t \geq 0$, we can imagine wrapping a line segment around the unit circle, marking off a distance of t in the counterclockwise direction, and labeling that endpoint $P(x, y)$, which becomes the terminal point. If $t < 0$, we would wrap in the clockwise direction.



If $t > 2\pi$ or $t < -2\pi$, then the length is longer than the circumference of the unit circle and we will need to travel around the unit circle more than once before arriving at the point $P(x, y)$. Therefore, we can conclude that regardless of the value of t, we have a unique point $P(x, y)$ that lies on the unit circle. We call $P(x, y)$ the *point on the unit circle that corresponds to* t.

1.3.2 Trigonometric Functions

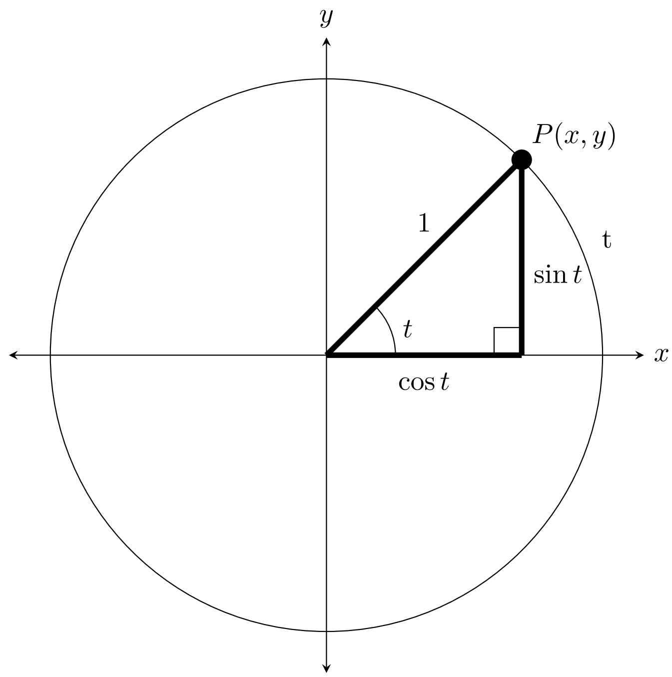
The x- and y-coordinates for $P(x, y)$ can then be used to define the six trigonometric functions of a real number t:

sine *cosine* *tangent* *cosecant* *secant* *cotangent*

which are abbreviated as *sin*, *cos*, *tan*, *csc*, *sec*, and *cot*, respectively.

Definition 1.3.2 Definition of Trigonometric Functions. Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle corresponding with t. Then

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x},\ (x \neq 0)$$
$$\csc t = \frac{1}{y},\ (y \neq 0) \qquad \sec t = \frac{1}{x},\ (x \neq 0) \qquad \cot t = \frac{x}{y},\ (y \neq 0).$$



Notice that $\tan t$ and $\sec t$ are undefined when $x = 0$ and $\csc t$ and $\cot t$ are undefined when $y = 0$. ♢

Example 1.3.3 Let t be the angle that corresponds to the point $P(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Find the exact values of the six trigonometric functions corresponding to t: $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, $\cot t$.

Solution. The point $P(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ gives us $x = \frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$. Then we have

$$\sin\theta = y = -\frac{1}{2}, \qquad \csc\theta = \frac{1}{y} = \frac{1}{-\frac{1}{2}} = -2,$$

$$\cos\theta = x = \frac{\sqrt{3}}{2}, \qquad \sec\theta = \frac{1}{x} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}},$$

$$\tan\theta = \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}, \qquad \cot\theta = \frac{x}{y} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}.$$

□

1.3.3 Trigonometric Functions of an Angle

Definition 1.3.4 Trigonometric Functions of an Angle. If θ is an angle with radian measure t, then the **six trigonometric functions** become

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x},\ (x \neq 0)$$

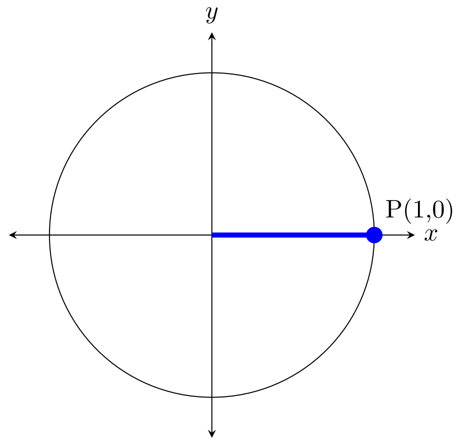
$$\csc\theta = \frac{1}{y},\ (y \neq 0) \qquad \sec\theta = \frac{1}{x},\ (x \neq 0) \qquad \cot\theta = \frac{x}{y},\ (y \neq 0).$$

♢

Example 1.3.5 Find the exact values of the six trigonometric functions for:

(a) $\theta = 0$

Solution. When $\theta = 0$ radians ($0°$), the point on the circle is $P(1, 0)$.

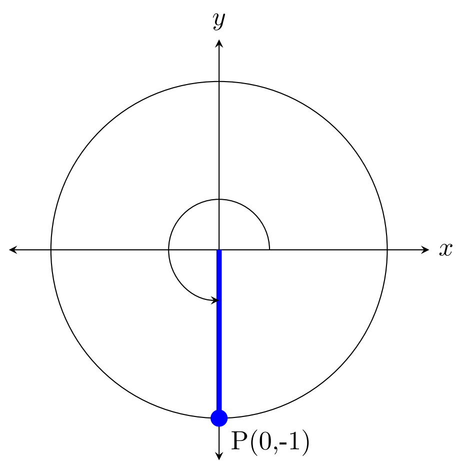


Then $x = 1$ and $y = 0$ gives us

$$\begin{aligned} \sin 0 &= \sin 0^\circ = 0, & \csc 0 &= \csc 0^\circ = \text{undefined}, \\ \cos 0 &= \cos 0^\circ = 1, & \sec 0 &= \sec 0^\circ = 1, \\ \tan 0 &= \tan 0^\circ = 0, & \cot 0 &= \cot 0^\circ = \text{undefined}. \end{aligned}$$

(b) $\theta = \frac{3\pi}{2}$

Solution. When $\theta = \frac{3\pi}{2}$ radians ($270°$), the point on the circle is $P(0, -1)$.



Then $x = 0$ and $y = -1$ gives us

$$\begin{aligned} \sin \frac{3\pi}{2} &= \sin 270^\circ = -1, & \csc \frac{3\pi}{2} &= \csc 270^\circ = -1, \\ \cos \frac{3\pi}{2} &= \cos 270^\circ = 0, & \sec \frac{3\pi}{2} &= \sec 270^\circ = \text{undefined}, \\ \tan \frac{3\pi}{2} &= \tan 270^\circ = \text{undefined}, & \cot \frac{3\pi}{2} &= \cot 270^\circ = 0. \end{aligned}$$

(c) $\theta = 5\pi$

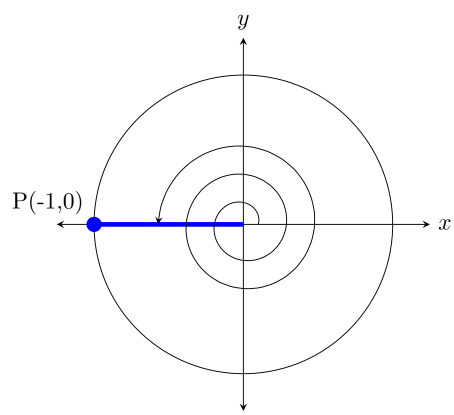
Solution. Since $\theta = 5\pi > 2\pi$, our angle is greater than one full rotation of a circle. We first subtract θ by one rotation, 2π, to get

$$5\pi - 2\pi - = 3\pi$$

Once again, since we have completed more than one full rotation, we can repeat the previous step:

$$3\pi - 2\pi = \pi$$

The values of the six trigonometric functions when $\theta = 5\pi$ are equal to those when $\theta = \pi$. Notice that 5π and π are *coterminal angles*, both ending at the point$P(-1, 0)$.



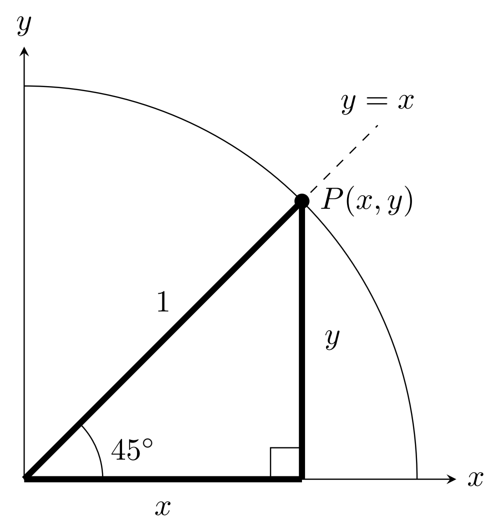
Since $x = -1$ and $y = 0$ we have

$$\begin{array}{lll} \sin 5\pi = 0, & \cos 5\pi = -1, & \tan 5\pi = 0, \\ \csc 5\pi = \text{undefined}, & \sec 5\pi = -1, & \cot 5\pi = \text{undefined}. \end{array}$$

□

Example 1.3.6 Finding the Exact Values of the Trigonometric Functions for $\theta = 45°$. Find the exact values of the six trigonometric functions for $\theta = 45°$.

Solution. We begin by drawing a right triangle with a base angle of 45° in the unit circle.



Since the first quadrant has 90°, at $\theta = 45°$, the point P lies on the line that bisects the first quadrant. This means the point P is on the line $y = x$. Since $P(x, y)$ also lies on the unit circle, whose equation is $x^2 + y^2 = 1$, we get

$$x^2 + y^2 = 1$$

$$
\begin{aligned}
x^2 + x^2 &= 1 && \text{(since } y = x\text{)} \\
2x^2 &= 1 \\
x^2 &= \frac{1}{2} \\
x &= \frac{1}{\sqrt{2}} \\
y &= \frac{1}{\sqrt{2}} && \text{(since } y = x\text{).}
\end{aligned}
$$

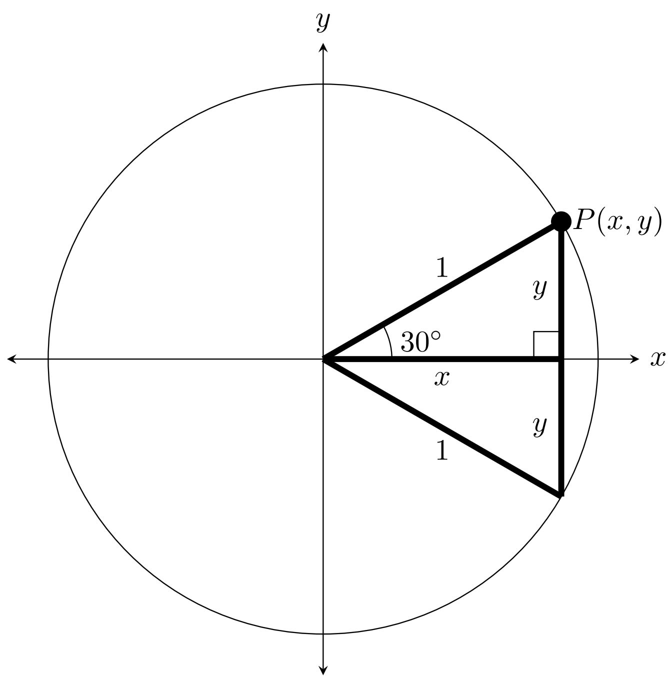
Then

$$
\begin{aligned}
\sin 45^\circ &= \frac{1}{\sqrt{2}}, & \csc 45^\circ &= \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}, \\
\cos 45^\circ &= \frac{1}{\sqrt{2}}, & \sec 45^\circ &= \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}, \\
\tan 45^\circ &= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1, & \cot 45^\circ &= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1.
\end{aligned}
$$

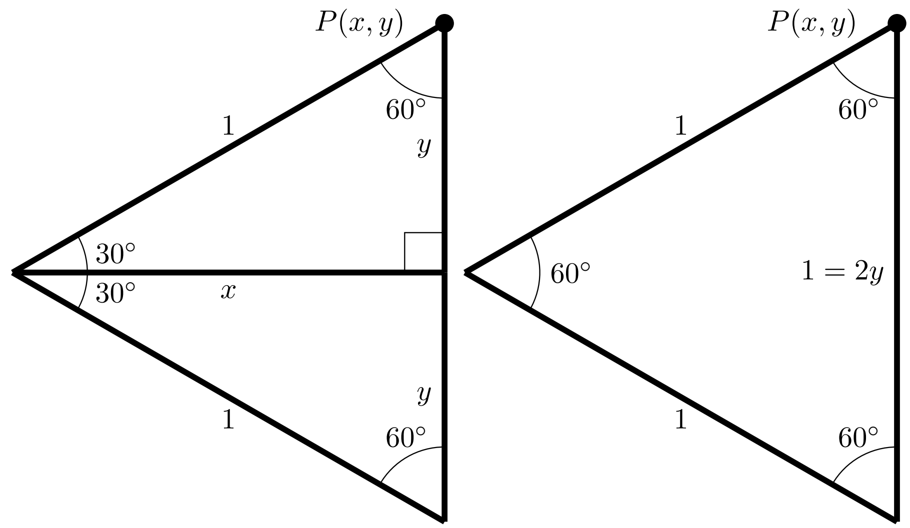
□

Example 1.3.7 Finding the Exact Values of the Trigonometric Functions for $\theta = 30°$. Find the exact values of the six trigonometric functions for $\theta = 30°$.

Solution. First, we will draw a triangle in a circle with an angle of 30° and a second triangle with an angle of −30°.



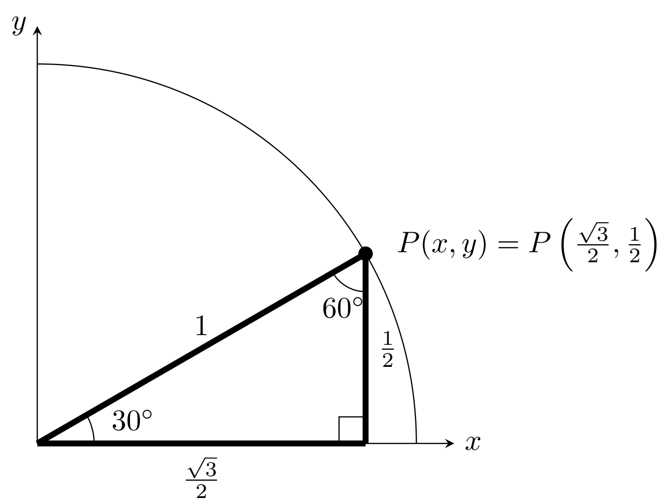
This gives us two 30-60-90 triangles. These two triangles give us one larger triangle whose angles are all 60°. Thus we have an equilateral triangle, with each side of length 1.



We see that $1 = 2y$ so $y = \frac{1}{2}$. Then by the Pythagorean Theorem,

$$\begin{aligned} x^2 + y^2 &= 1^2 \\ x^2 + \left(\frac{1}{2}\right)^2 &= 1 \\ x^2 + \frac{1}{4} &= 1 \\ x^2 &= \frac{3}{4} \\ x &= \frac{\sqrt{3}}{2}, \end{aligned}$$

giving us the following triangle.



Then

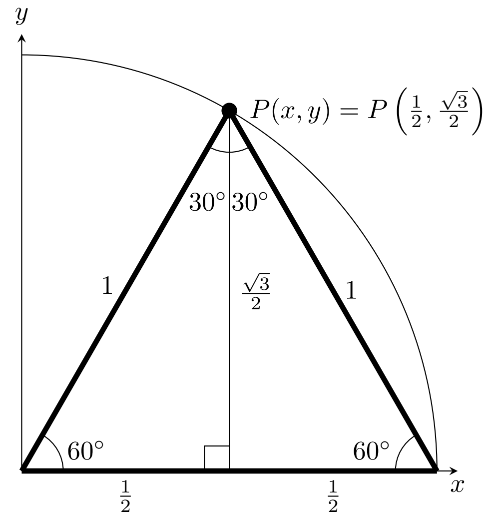
$$\sin 30° = \frac{1}{2}, \qquad \csc 30° = \frac{1}{\frac{1}{2}} = 2,$$

$$\cos 30° = \frac{\sqrt{3}}{2}, \qquad \sec 30° = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}},$$

$$\tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}, \qquad \cot 30^\circ = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

□

Remark 1.3.8 Finding the Exact Values of the Trigonometric Functions for $\theta = 90°$. Similarly, we can get the following for $\theta = 60°$.



We now summarize what we know about the six trigonometric functions for special angles. Note the trigonometric functions for $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{3}$ are left as exercises.

Table 1.3.9 Trigonometric functions for special angles (Undefined values are abbreviated as "undef").

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	0	1	0	undef	1	undef
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
90°	$\frac{\pi}{2}$	1	0	undef	1	undef	0

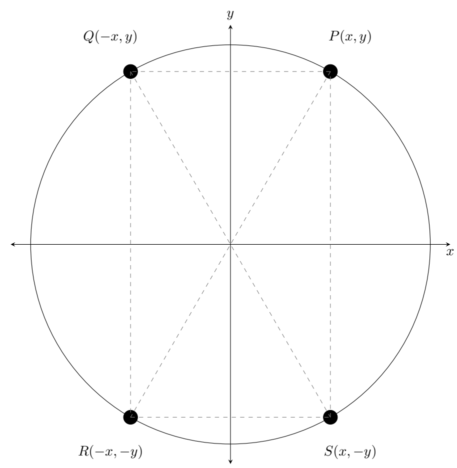
1.3.4 Symmetry on the Unit Circle

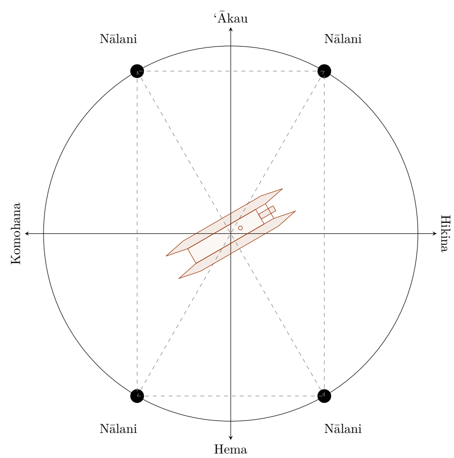
If the point $P(x, y)$ lies on the unit circle, the following symmetric points also lie on the unit circle:

1. $Q(-x, y)$: Symmetry about the y-axis.

2. $R(-x,-y)$: Symmetry about the origin.
3. $S(x,-y)$: Symmetry about the x-axis.

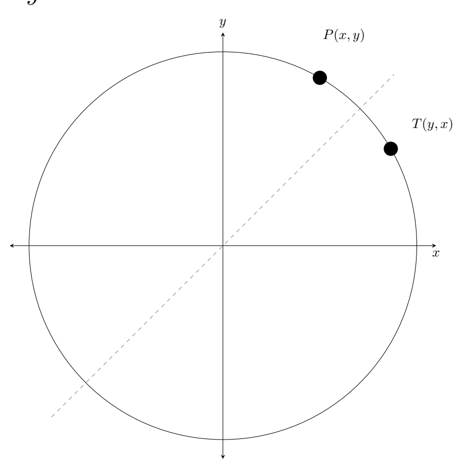
This symmetry within the unit circle resembles the pattern observed in the Star Compass. When a star emerges in the eastern sky, it will eventually descend and set in the corresponding house of the western sky. For instance, if a star rises above the horizon in the Nālani house of the Koʻolau quadrant (northeast), it will journey across the sky and set in the equivalent house within the Hoʻolua quadrant (northwest). This similarity aligns with the symmetry between points $P(x,y)$ and $Q(-x,y)$. Additionally, if an ocean swell or wind originates from the Nālani house in the Malanai quadrant (southeast), it will pass the waʻa and exit in the opposite direction toward the Hoʻolua quadrant (northwest), still within the Nālani house. This mirrors the symmetry between points $S(x,-y)$ and $Q(-x,y)$.



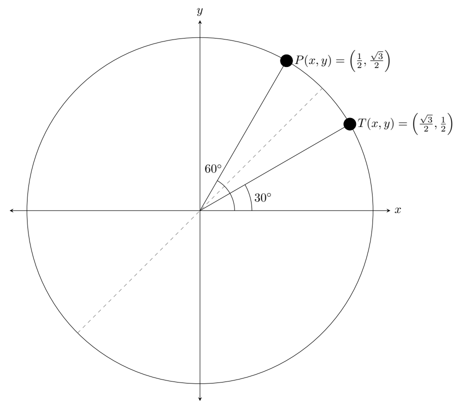


A fourth form of symmetry involves reflecting points across the diagonal line $y=x$, where the x- and y-values are equal.

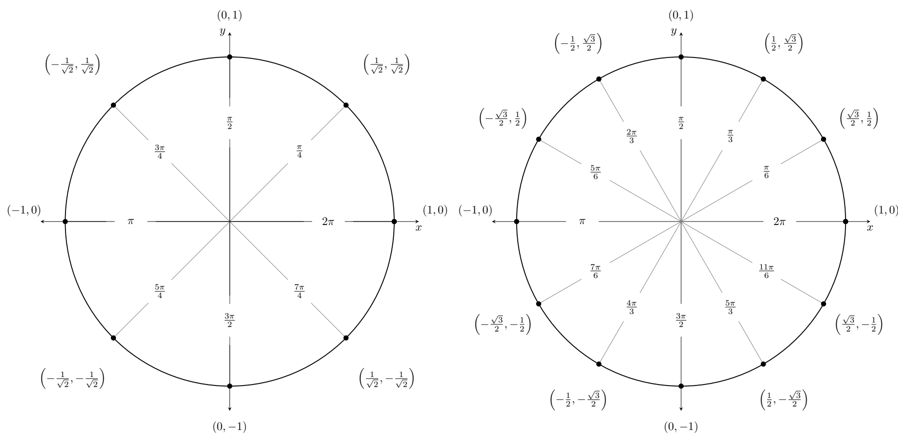
$T(y,x)$: Symmetry about the line $y=x$. This is accomplished by interchanging the x- and y-values.



Notice on the Unit Circle that the radius extending from the center at an angle of 30° to the point $T(x,y)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ is symmetric about the line $y=x$, in relation to the radius extending from the center at an angle of 60° to the point $P(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.



Using symmetry about the x-axis, symmetry about the y-axis, and symmetry about the origin, we can complete the unit circle, as long as we remember that the x-values in Quadrants II and III are negative while the y-values in Quadrants III and IV are negative.



Finally, we tie everything together and look at the entire Unit Circle. At first glance, it may seem intimidating; however, similar to the Star Compass, there is a lot of symmetry (x-axis, y-axis, origin, and the line $y = x$). A helpful approach is to focus on one quadrant and use symmetry to complete the rest of the circle.

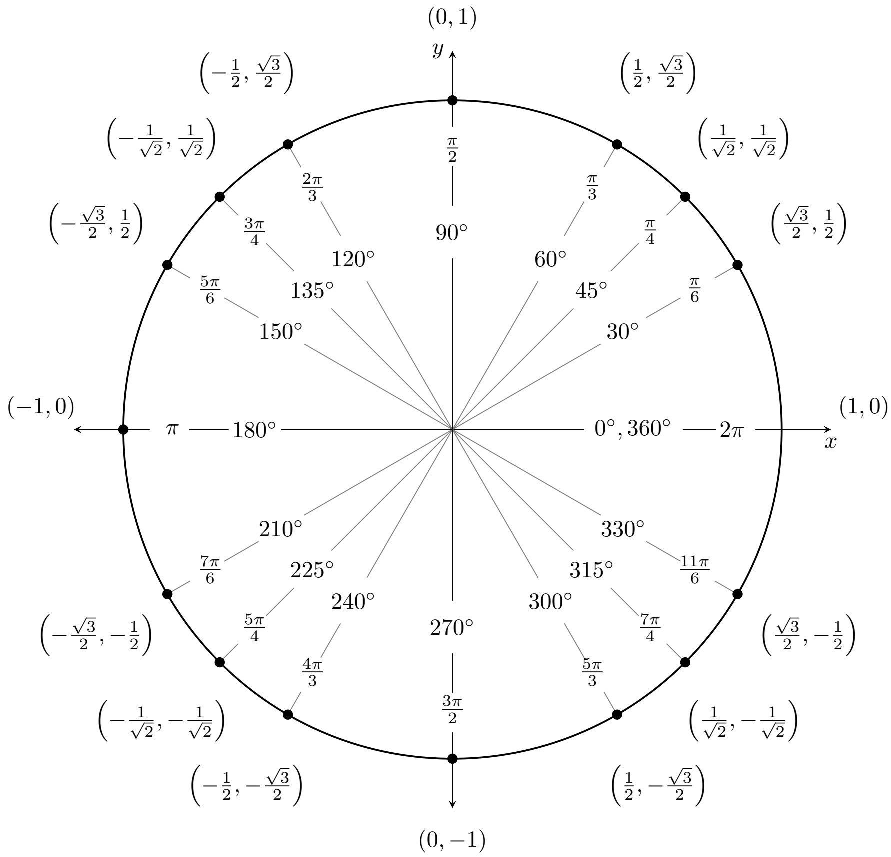


Figure 1.3.10 The Unit Circle for common angles in radians and degrees.

1.3.5 Trigonometric Functions on a Circle with Radius r

Until now, computing the exact values of trigonometric functions of an angle θ required us to locate the corresponding point $P(x, y)$ on the unit circle. However, we can use any circle with center at the origin, that is, any circle of the form $x^2 + y^2 = r^2$, where $r > 0$ is the radius. Note that if $r = 1$, then the cirlce is the unit circle.

Theorem 1.3.11 *For an angle θ in standard position, let $P(x, y)$ be the point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. Then*

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ (x \neq 0)$$
$$\csc\theta = \frac{r}{y},\ (y \neq 0) \qquad \sec\theta = \frac{r}{x},\ (x \neq 0) \qquad \cot\theta = \frac{x}{y},\ (y \neq 0).$$

1.3.6 Exercises

Exercise Group. Verify algebraically that the point P is on the unit circle $(x^2 + y^2 = 1)$.

1. $P\left(\frac{3}{5}, -\frac{4}{5}\right)$

Answer. $\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = 1$

2. $P\left(-\frac{\sqrt{39}}{8}, -\frac{5}{8}\right)$

Answer. $\left(-\frac{\sqrt{39}}{8}\right)^2 + \left(-\frac{5}{8}\right)^2 = 1$

3. $P\left(-\frac{\sqrt{55}}{8}, \frac{3}{8}\right)$

Answer. $\left(-\frac{\sqrt{55}}{8}\right)^2 + \left(\frac{3}{8}\right)^2 = 1$

4. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

Answer. $\left(-\frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2 = 1$

5. $P\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

Answer. $\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2 = 1$

6. $P\left(\frac{\sqrt{21}}{5}, -\frac{2}{5}\right)$

Answer. $\left(\frac{\sqrt{21}}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 = 1$

Exercise Group. Let the point P be on the unit circle. Given the quadrant that P lies in, determine the missing coordinate, a.

7. III; $P\left(-\frac{2}{3}, a\right)$

Answer. $-\frac{\sqrt{5}}{3}$

8. IV; $P\left(\frac{5}{8}, a\right)$

Answer. $-\frac{\sqrt{39}}{8}$

9. III; $P\left(a, -\frac{2}{5}\right)$

Answer. $-\frac{\sqrt{21}}{5}$

10. II; $P\left(a, \frac{4}{9}\right)$

Answer. $-\frac{\sqrt{65}}{9}$

Exercise Group. Given an angle θ that corresponds to the point P on the unit circle, determine the coordinates of the point $P(x, y)$.

11. $\theta = \frac{\pi}{2}$

Answer. $(0, 1)$

12. $\theta = \pi$

Answer. $(-1, 0)$

13. $\theta = \frac{5\pi}{3}$

Answer. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

14. $\theta = \frac{4\pi}{3}$

Answer. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

15. $\theta = -\frac{\pi}{4}$

Answer. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

16. $\theta = \frac{5\pi}{6}$

Answer. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

17. $\theta = 315°$

Answer. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

18. $\theta = 720°$

Answer. $(1, 0)$

19. $\theta = 60°$

Answer. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

20. $\theta = -180°$

Answer. $(-1, 0)$

21. $\theta = 210°$

Answer. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

22. $\theta = 120°$

Answer. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Exercise Group. For each angle θ in Exercises 1.3.6.11–22, find the exact values of the six trigonometric functions. If any are not defined, say "*undefined*."

23. $\theta = \frac{\pi}{2}$

Answer. $\sin\frac{\pi}{2} = 1$; $\cos\frac{\pi}{2} = 0$; $\tan\frac{\pi}{2}$ is undefined; $\csc\frac{\pi}{2} = 1$; $\sec\frac{\pi}{2}$ is undefined; $\cot\frac{\pi}{2} = 0$

24. $\theta = \pi$

Answer. $\sin\pi = 0$; $\cos\pi = -1$; $\tan\pi = 0$; $\csc\pi$ is undefined; $\sec\pi = -1$; $\cot\pi$ is undefined

25. $\theta = \frac{5\pi}{3}$

Answer. $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$; $\cos\frac{5\pi}{3} = \frac{1}{2}$; $\tan\frac{5\pi}{3} = -\sqrt{3}$; $\csc\frac{5\pi}{3} = -\frac{2}{\sqrt{3}}$; $\sec\frac{5\pi}{3} = 2$; $\cot\frac{5\pi}{3} = -\frac{1}{\sqrt{3}}$

26. $\theta = \frac{4\pi}{3}$

Answer. $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$; $\cos\frac{4\pi}{3} = -\frac{1}{2}$; $\tan\frac{4\pi}{3} = \sqrt{3}$; $\csc\frac{4\pi}{3} = -\frac{2}{\sqrt{3}}$; $\sec\frac{4\pi}{3} = -2$; $\cot\frac{4\pi}{3} = \frac{1}{\sqrt{3}}$

27. $\theta = -\frac{\pi}{4}$

Answer. $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$; $\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $\tan\left(-\frac{\pi}{4}\right) = -1$; $\csc\left(-\frac{\pi}{4}\right) = -\sqrt{2}$; $\sec\left(-\frac{\pi}{4}\right) = \sqrt{2}$; $\cot\left(-\frac{\pi}{4}\right) = -1$

28. $\theta = \frac{5\pi}{6}$

Answer. $\sin\frac{5\pi}{6} = \frac{1}{2}$; $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$; $\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$; $\csc\frac{5\pi}{6} = 2$; $\sec\frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$; $\cot\frac{5\pi}{6} = -\sqrt{3}$

29. $\theta = 315°$

Answer. $\sin 315° = -\frac{1}{\sqrt{2}}$; $\cos 315° = \frac{1}{\sqrt{2}}$; $\tan 315° = -1$; $\csc 315° = -\sqrt{2}$; $\sec 315° = \sqrt{2}$; $\cot 315° = -1$

30. $\theta = 720°$

Answer. $\sin 720° = 0$; $\cos 720° = 1$; $\tan 720° = 0$; $\csc 720°$ is undefined; $\sec 720° = 1$; $\cot 720°$ is undefined

31. $\theta = 60°$

Answer. $\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \sqrt{3}$; $\csc 60° = \frac{2}{\sqrt{3}}$; $\sec 60° = 2$; $\cot 60° = \frac{1}{\sqrt{3}}$

32. $\theta = -180°$

Answer. $\sin(-180°) = 0$; $\cos(-180°) = -1$; $\tan(-180°) = 0$; $\csc(-180°)$ is undefined; $\sec(-180°) = -1$; $\cot(-180°)$ is undefined

33. $\theta = 210°$

Answer. $\sin 210° = -\frac{1}{2}$; $\cos 210° = -\frac{\sqrt{3}}{2}$; $\tan 210° = \frac{1}{\sqrt{3}}$; $\csc 210° = -2$; $\sec 210° = -\frac{2}{\sqrt{3}}$; $\cot 210° = \sqrt{3}$

34. $\theta = 120°$

Answer. $\sin 120° = \frac{\sqrt{3}}{2}$; $\cos 120° = -\frac{1}{2}$; $\tan 120° = -\sqrt{3}$; $\csc 120° = \frac{2}{\sqrt{3}}$; $\sec 120° = -2$; $\cot 120° = -\frac{1}{\sqrt{3}}$

Exercise Group. Let θ be the angle that corresponds to the point P. Exercises 1.3.6.1–6 verified P is on the unit circle. Find the exact values of the six trigonometric functions of θ.

35. $P\left(\frac{3}{5}, -\frac{4}{5}\right)$

Answer. $\sin\theta = -\frac{4}{5}$; $\cos\theta = \frac{3}{5}$; $\tan\theta = -\frac{4}{3}$; $\csc\theta = -\frac{5}{4}$; $\sec\theta = \frac{5}{3}$; $\cot\theta = -\frac{3}{4}$

36. $P\left(-\frac{\sqrt{39}}{8}, -\frac{5}{8}\right)$

Answer. $\sin\theta = -\frac{5}{8}$; $\cos\theta = -\frac{\sqrt{39}}{8}$; $\tan\theta = \frac{5}{\sqrt{39}}$; $\csc\theta = -\frac{8}{5}$; $\sec\theta = -\frac{8}{\sqrt{39}}$; $\cot\theta = \frac{\sqrt{39}}{5}$

37. $P\left(-\frac{\sqrt{55}}{8}, \frac{3}{8}\right)$

Answer. $\sin\theta = \frac{3}{8}$; $\cos\theta = -\frac{\sqrt{55}}{8}$; $\tan\theta = -\frac{3}{\sqrt{55}}$; $\csc\theta = \frac{8}{3}$; $\sec\theta = -\frac{8}{\sqrt{55}}$; $\cot\theta = -\frac{\sqrt{55}}{3}$

38. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

Answer. $\sin\theta = \frac{\sqrt{5}}{3}$; $\cos\theta = -\frac{2}{3}$; $\tan\theta = -\frac{\sqrt{5}}{2}$; $\csc\theta = \frac{3}{\sqrt{5}}$; $\sec\theta = -\frac{3}{2}$; $\cot\theta = -\frac{2}{\sqrt{5}}$

39. $P\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

Answer. $\sin\theta = \frac{\sqrt{7}}{4}$; $\cos\theta = \frac{3}{4}$; $\tan\theta = \frac{\sqrt{7}}{3}$; $\csc\theta = \frac{4}{\sqrt{7}}$; $\sec\theta = \frac{4}{3}$; $\cot\theta = \frac{3}{\sqrt{7}}$

40. $P\left(\frac{\sqrt{21}}{5}, -\frac{2}{5}\right)$

Answer. $\sin\theta = -\frac{2}{5}$; $\cos\theta = \frac{\sqrt{21}}{5}$; $\tan\theta = -\frac{2}{\sqrt{21}}$; $\csc\theta = -\frac{5}{2}$; $\sec\theta = \frac{5}{\sqrt{21}}$; $\cot\theta = -\frac{\sqrt{21}}{2}$

Exercise Group. Find the exact value of each expression.

41. $\sin 30° + \sin 150°$

Answer. 1

42. $\cos 30° + \cos 150°$

Answer. 0

43. $\sin 60° + \sin 120° + \sin 240° + \sin 300°$

Answer. 0

44. $\cos 60° + \cos 120° + \cos 240° + \cos 300°$

Answer. 0

45. $\tan 45^\circ + \tan 135^\circ$

Answer. 0

46. $\tan 135^\circ + \tan 225^\circ$

Answer. 0

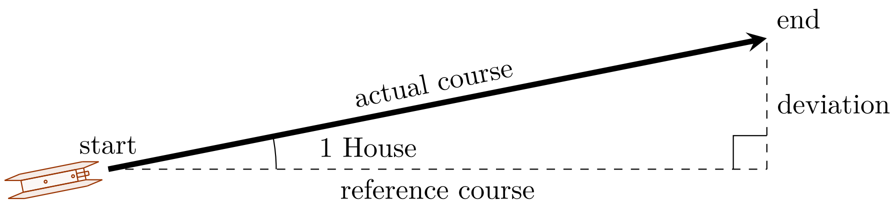
47. $\tan 225^\circ + \tan 315^\circ$

Answer. 0

48. $\tan 45^\circ + \tan 225^\circ$

Answer. 2

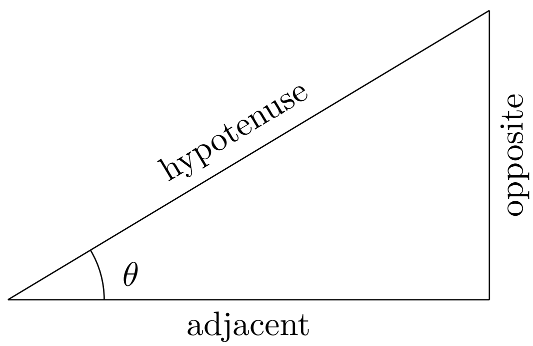
1.4 Right Triangle Trigonometry



During a voyage, a navigator utilizes a *reference course* —a line connecting the starting point and destination—to monitor their position. When the wa‘a (canoe) encounters winds that veer it off course, the navigator mentally plots their position relative to the reference course. To ensure the destination isn't missed, navigators must monitor their deviation from the intended course, involving measurement of the angle of deviation from the reference course (in units of houses) and determining the distance traveled. This section explores the calculation of trigonometric functions using right triangles, enabling us to assess how much the wa‘a has strayed from its intended reference course.

1.4.1 Trigonometric Ratios

Definition 1.4.1 Trigonometric Ratios. Consider a right triangle with θ as one of its acute angles. The trigonometric ratios are defined as follows:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}},$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}, \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}.$$

A common mnemonic for remembering these relationships is SOHCAHTOA, formed from the first letters of "*S*ine is *O*pposite over *H*ypotenuse, *C*osine is *A*djacent over *H*ypotenuse, *T*angent is *O*pposite over *A*djacent." ◊

Based on the definition of the six trigonometric functions, we have the following trigonometric identities.

Definition 1.4.2 Reciprocal Identities.

$$\sin\theta = \frac{1}{\csc\theta}, \qquad \cos\theta = \frac{1}{\sec\theta}, \qquad \tan\theta = \frac{1}{\cot\theta},$$
$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

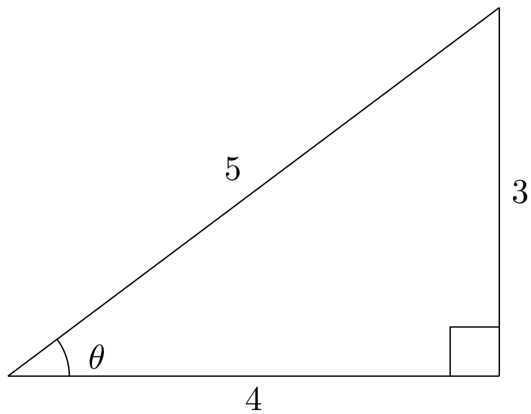
◊

Definition 1.4.3 Quotient Identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

◊

Example 1.4.4 Find the exact values of the six trigonometric ratios of the angle θ in the given triangle.

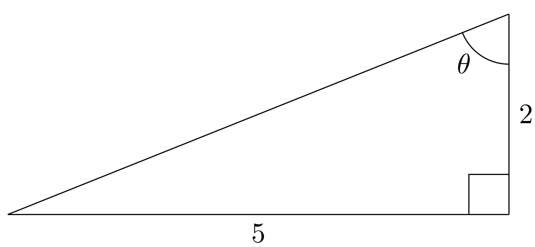


Solution. By the definition of the trigonometric ratios, we have

$$\sin\theta = \frac{3}{5}, \qquad \cos\theta = \frac{4}{5}, \qquad \tan\theta = \frac{3}{4},$$
$$\csc\theta = \frac{5}{3}, \qquad \sec\theta = \frac{5}{4}, \qquad \cot\theta = \frac{4}{3}.$$

□

Example 1.4.5 Find the exact values of the six trigonometric ratios of the angle θ in the given triangle.



Solution. Notice that θ is in a different position. Here, the adjacent side is three and the opposite side is five. If we let h denote the hypotenuse, then we can use the Pythagorean Theorem to get

$$h = \sqrt{5^2 + 2^2} = \sqrt{29}.$$

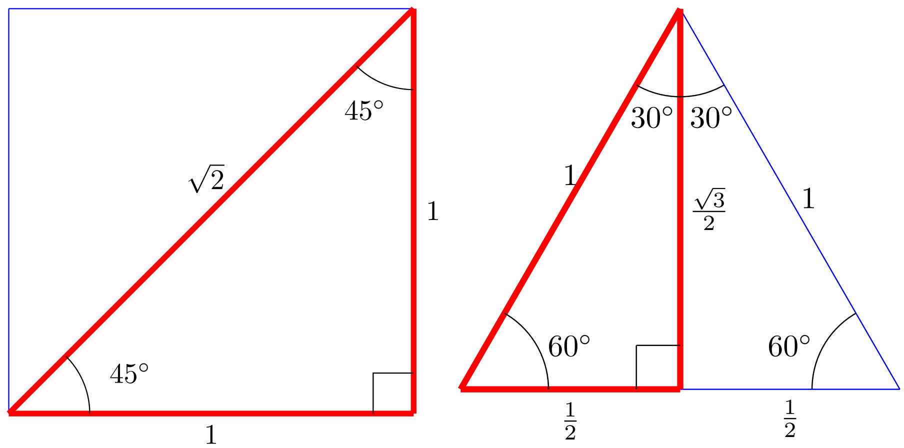
Then by the definition of the trigonometric ratios, we have

$$\sin\theta = \frac{5}{\sqrt{29}}, \qquad \cos\theta = \frac{2}{\sqrt{29}}, \qquad \tan\theta = \frac{5}{2},$$
$$\csc\theta = \frac{\sqrt{29}}{5}, \qquad \sec\theta = \frac{\sqrt{29}}{2}, \qquad \cot\theta = \frac{2}{5}.$$

□

1.4.2 Special Triangles

The angles $30°, 45°, 60°$ $(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3})$ give special values for trigonometric functions. The following figures are used to calculate trigonometric values.



The trigonometric values for the special angles $0°, 30°, 45°, 60°, 90°$ $\left(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ are given in Table 1.4.6.

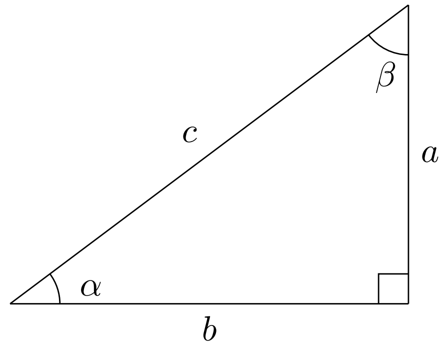
Table 1.4.6 Values of the trigonometric functions in Quadrant I

θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined

1.4.3 Cofunctions

The symmetry between $\sin\theta$ and $\cos\theta$ becomes evident when reversing the order of sine and cosine values from $90°$ to $0°$. This symmetry yields $\sin 0° = \cos 90°$, $\sin 30° = \cos 60°$, $\sin 45° = \cos 45°$, $\sin 60° = \cos 30°$, and $\sin 90° = \cos 0°$.

This pattern between sine and cosine is no coincidence; it emerges because the three angles in a triangle add up to $180°$ or π radians. When considering a right triangle, the remaining two angles combine to form $90°$ or $\frac{\pi}{2}$ radians, making them *complementary angles*.



Consider the right triangle in the figure above, where angles α and β are complementary angles. Side a is opposite of angle α, and side b is opposite of angle β. Notice that we can also describe side b as adjacent to angle α and side a as adjacent to angle β. Therefore,

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \text{ and } \cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}.$$

Thus we can conclude that

$$\sin\alpha = \frac{a}{c} = \cos\beta.$$

Sine and cosine are called **cofunctions** because of this relationship between these functions and their complementary angles. We can obtain similar relationships for all trigonometric functions:

$$\begin{aligned} \sin\alpha &= \frac{a}{c} = \cos\beta, & \cos\alpha &= \frac{b}{c} = \sin\beta, & \tan\alpha &= \frac{a}{b} = \cot\beta, \\ \csc\alpha &= \frac{c}{a} = \sec\beta, & \sec\alpha &= \frac{c}{b} = \csc\beta, & \cot\alpha &= \frac{b}{a} = \tan\beta. \end{aligned}$$

Since α and β are complementary angles, $\alpha+\beta = 90°$. Rearranging, we get $\beta = 90° - \alpha$. Substituting this into our cofunctions and replacing α with θ, we get our **cofunction identities**.

Definition 1.4.7 Cofunction Identities. The cofunction identities in degrees are

$$\begin{aligned} \sin\theta &= \cos(90° - \theta), & \cos\theta &= \sin(90° - \theta), & \tan\theta &= \cot(90° - \theta), \\ \csc\theta &= \sec(90° - \theta), & \sec\theta &= \csc(90° - \theta), & \cot\theta &= \tan(90° - \theta). \end{aligned}$$

The cofunction identities in radians are

$$\begin{aligned} \sin\theta &= \cos\left(\frac{\pi}{2} - \theta\right), & \cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right), & \tan\theta &= \cot\left(\frac{\pi}{2} - \theta\right), \\ \csc\theta &= \sec\left(\frac{\pi}{2} - \theta\right), & \sec\theta &= \csc\left(\frac{\pi}{2} - \theta\right), & \cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right). \end{aligned}$$

◊

Example 1.4.8 Cofunction Identities explain the symmetry in Table 1.4.6.

$$\begin{aligned} \sin 0° &= \cos(90° - 0°) = \cos 90°, & \sin 30° &= \cos(90° - 30°) = \cos 60°, \\ \sin 45° &= \cos(90° - 45°) = \cos 45°, & \sin 60° &= \cos(90° - 60°) = \cos 30°, \\ \sin 90° &= \cos(90° - 90°) = \cos 0°. \end{aligned}$$

□

Remark 1.4.9 Patterns in the Trigonometric Table. To help remember the values of sine and cosine, we utilize cofunctions and also write them in the form $\sqrt{\cdot}/2$.

θ	θ	$\sin\theta$	$\cos\theta$
0	$0°$	$\sqrt{0}/2$	$\sqrt{4}/2$
$\pi/6$	$30°$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/4$	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$60°$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/2$	$90°$	$\sqrt{4}/2$	$\sqrt{0}/2$

which simplifies to the values in Table 1.4.6.

1.4.4 Using a Calculator

Sometimes we may encounter an angle other than the special angles described above. In this case, we will have to use a calculator.

First, ensure that the angle is either in degrees or radians, depending on the problem. Refer to your calculator's manual for instructions. Most calculators have dedicated buttons for the sine, cosine, and tangent functions. Depending on your calculator, you may see the following keys

Function	Calculator Key
sine	*SIN*
cosine	*COS*
tangent	*TAN*

To calculate cosecant, secant, and cotangent, we will need to use the identities

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Answers produced by calculators are *estimates* and we should pay close attention to see if the question is asking for the exact solution or a decimal approximation. For example, if we need to calculate $\sin 45° = \frac{1}{\sqrt{2}}$, the calculator may give the answer as $\sin 45° \approx 0.70710678$, which is a decimal approximation since the actual value is an irrational number with infinitely many decimal places. *Unless stated otherwise, answers in the book should be exact*, e.g. $\frac{1}{\sqrt{2}}$ and not 0.70710678.

Example 1.4.10 Use a calculator to evaluate:

(a) $\sin 22°$

Solution. Before proceeding, we confirm that our calculator is set to either degree or radian mode. Additionally, for the sake of simplicity, we will round our answers to four decimal places.

Input: *SIN* (22); Output: 0.3746

(b) $\cos 5°$

Solution. Input: *COS* (5); Output: 0.9962

(c) $\cot 53°$

Solution. Since most calculators do not have a key for cotangent, we Input: (1/ *TAN* (53)); Output: $\dfrac{1}{1.3270} \approx 0.7536$

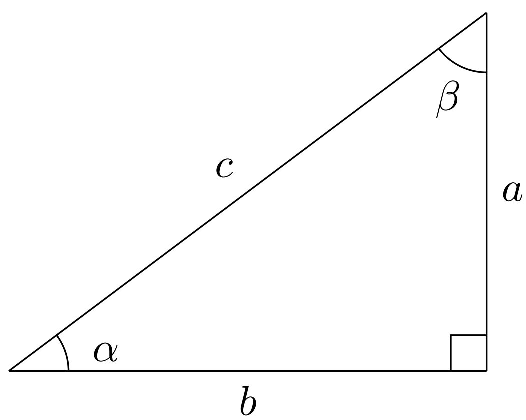
(d) $\cos 5$ rad

Solution. Since this problem uses radians, we must change the mode on our calculator then Input: *COS* (5); Output: 0.2837.

□

Remark 1.4.11 Observe that $\cos 5° \neq \cos 5$ rad. This emphasizes the significance of verifying whether the calculator is in degree or radian mode.

1.4.5 Solving Right Triangles



Consider the following right triangle where side a is opposite angle α, side b is opposite angle β, and side c is the hypotenuse. Since α and β are complementary angles, we have

$$\alpha + \beta = 90°.$$

Additionally, by the Pythagorean Theorem, we have

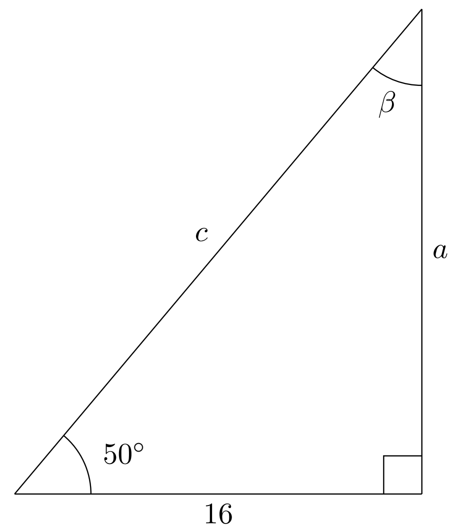
$$a^2 + b^2 = c^2.$$

Definition 1.4.12 To **solve a triangle** is the process of determining the values for all three lengths of its sides and the measures of all three angles, based on provided information about the triangle. ◊

Remark 1.4.13 Solving Right Triangles. In solving a right triangle, the following relationships are useful:

$$\alpha + \beta = 90°, \qquad a^2 + b^2 = c^2.$$

Example 1.4.14 Solve the right triangle. Round your answer to two decimal places.



Solution. Given that this is a right triangle, we already know one angle is $90°$, and we have an additional angle of $50°$ along with an adjacent side length of 16. To solve this triangle, we need to determine the values of sides a, c, and β. We begin by finding the measure of angle β. Since $50° + \beta = 90°$ we have

$$\beta = 90° - 50° = 40°.$$

Next, we will solve for side a. Using the angle $50°$, where the adjacent side is 16 and side a is the side opposite to the angle, we can apply the tangent function, which relates the opposite and adjacent sides:

$$\tan 50° = \frac{a}{16}.$$

Multiplying both sides by 16 we get

$$a = 16 \cdot \tan 50° \approx 19.07.$$

Using the Pythagorean Theorem, we get

$$c^2 = 16^2 + 19.07^2 \approx 619.66.$$

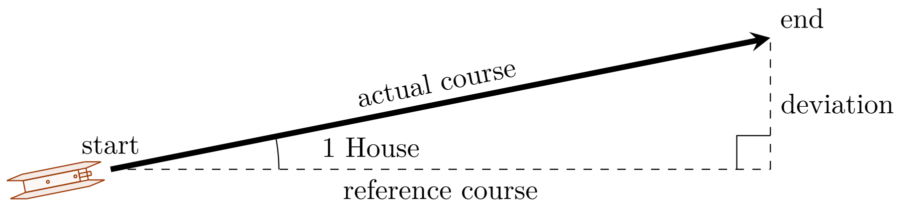
Thus

$$c \approx \sqrt{619.66} \approx 24.89.$$

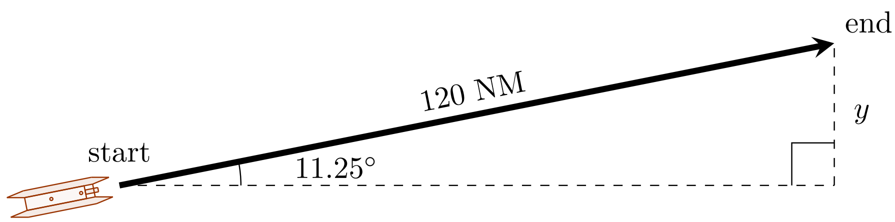
□

1.4.6 Solving Applied Problems

Example 1.4.15 Deviation. We are now ready to calculate the deviation example proposed at the start of this section. In an average day of sailing, a waʻa sails 120 nautical miles (NM). If Hikianalia is supposed to sail in the direction of Hikina (East), but currents have deviated her course by one house so she actually sailed in the house Lā, how far off the course has Hikianalia deviated?



Solution. From the Star Compass (Figure 1.1.4), the house Lā is one house (11.25°) from Hikina. If we let y denote the distance deviated from the reference course, our right triangle becomes:



Since we know the hypotenuse of the triangle and want to find the side opposite of the angle, we will use the sine function:

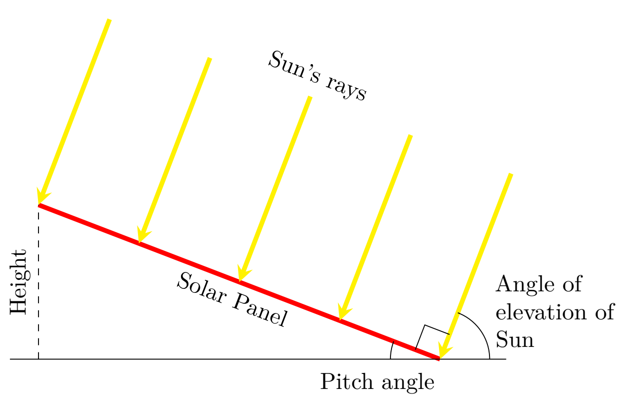
$$\sin 11.25^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{120 \text{ NM}};$$

multiplying both sides by 120, we get

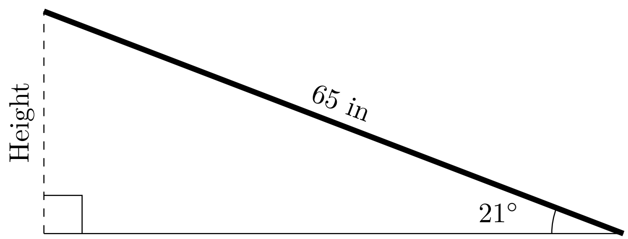
$$y = 120 \cdot \sin 11.25^\circ \text{ NM} \approx 23.4 \text{ NM}.$$

So Hikianalia has deviated 23.4 nautical miles north from the reference course. □

Example 1.4.16 Solar panels. Solar panels harness the sun's energy to generate electricity, and for optimal energy output, they should be oriented perpendicularly to the sun's light. The sun's angle of elevation varies based on latitude, and in Hawaiʻi, for instance, south-facing solar panels are recommended to have a pitch of 21° to align with the sun's rays. When installing a solar panel, determining its pitch might pose challenges. Instead of measuring the angle directly, an alternative approach involves measuring the height of the panel's top. What height should a south-facing solar panel, measuring 65 inches in length, be installed at to achieve the desired angle of 21°? Round your answer to the nearest tenth of an inch.



Solution. We begin by drawing the triangle.



Since we know the desired angle of pitch of the solar panel and the length of the panel, we can set up the following equation

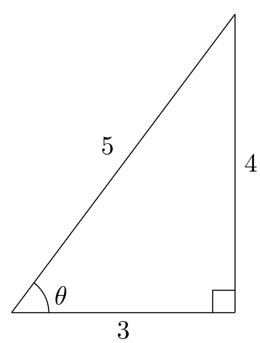
$$\sin 21^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{height}}{65 \text{ in}};$$
$$\text{height} = 65 \text{ in} \cdot \sin 21^\circ \approx 23.3 \text{ in}.$$

Thus, when installing a solar panel in Hawai‘i, the top of the solar panel should be positioned 23.3 inches above the bottom to optimize energy output. □

1.4.7 Exercises

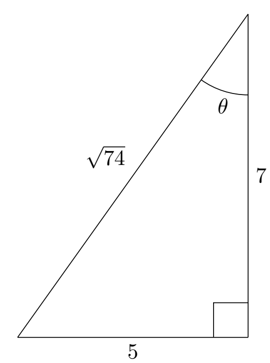
Exercise Group. Find the exact values of the six trigonometric functions of the angle θ in each triangle.

1.



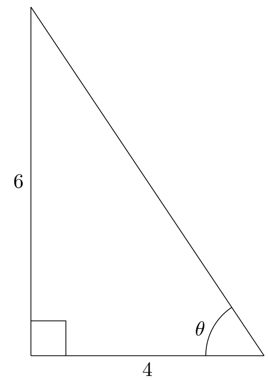
Answer. $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$

2.



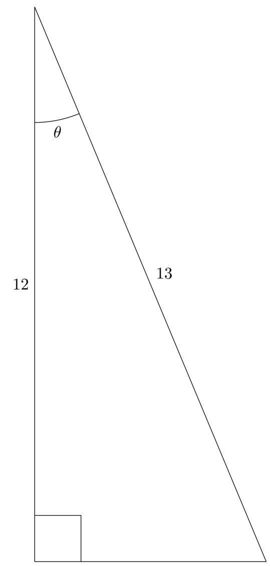
Answer. $\sin\theta = \frac{5}{\sqrt{74}}$, $\cos\theta = \frac{7}{\sqrt{74}}$, $\tan\theta = \frac{5}{7}$, $\csc\theta = \frac{\sqrt{74}}{5}$, $\sec\theta = \frac{\sqrt{74}}{7}$, $\cot\theta = \frac{7}{5}$

3.



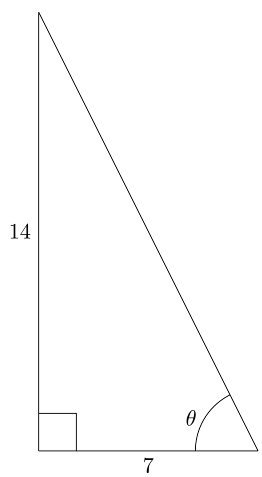
Answer. $\sin\theta = \frac{3}{\sqrt{13}}$, $\cos\theta = \frac{2}{\sqrt{13}}$, $\tan\theta = \frac{3}{2}$, $\csc\theta = \frac{\sqrt{13}}{3}$, $\sec\theta = \frac{\sqrt{13}}{2}$, $\cot\theta = \frac{2}{3}$

4.



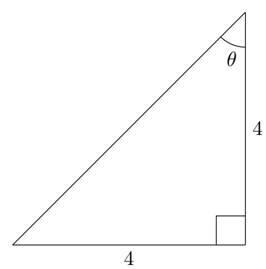
Answer. $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = \frac{12}{5}$

5.



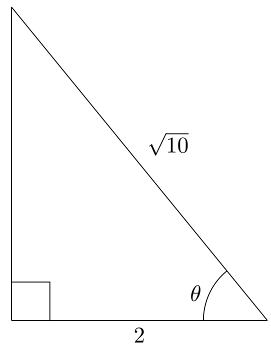
Answer. $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$, $\tan\theta = 2$, $\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = \sqrt{5}$, $\cot\theta = \frac{1}{2}$

6.



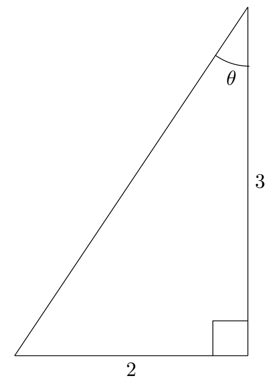
Answer. $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$, $\tan\theta = 1$, $\csc\theta = \sqrt{2}$, $\sec\theta = \sqrt{2}$, $\cot\theta = 1$

7.



Answer. $\sin\theta = \frac{\sqrt{3}}{\sqrt{5}}$,
$\cos\theta = \frac{2}{\sqrt{10}}$, $\tan\theta = \frac{\sqrt{6}}{2}$,
$\csc\theta = \frac{\sqrt{10}}{\sqrt{6}}$, $\sec\theta = \frac{\sqrt{10}}{2}$,
$\cot\theta = \frac{2}{\sqrt{6}}$

8.



Answer. $\sin\theta = \frac{2}{\sqrt{13}}$,
$\cos\theta = \frac{3}{\sqrt{13}}$, $\tan\theta = \frac{2}{3}$,
$\csc\theta = \frac{\sqrt{13}}{2}$, $\sec\theta = \frac{\sqrt{13}}{3}$,
$\cot\theta = \frac{3}{2}$

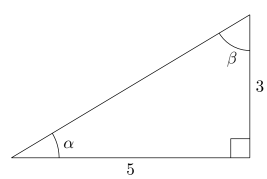
Exercise Group. For each of the following problems, calculate:

(a) $\cos\alpha$ and $\sin\beta$;

(b) $\tan\alpha$ and $\cot\beta$;

(c) $\csc\alpha$ and $\sec\beta$.

9.



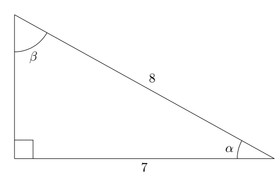
Answer.

(a) $\cos\alpha = \sin\beta = \frac{5}{\sqrt{34}}$;

(b) $\tan\alpha = \cot\beta = \frac{3}{5}$;

(c) $\csc\alpha = \sec\beta = \frac{\sqrt{34}}{3}$.

10.



Answer.

(a) $\cos\alpha = \sin\beta = \frac{7}{8}$;

(b) $\tan\alpha = \cot\beta = \frac{\sqrt{15}}{7}$;

(c) $\csc\alpha = \sec\beta = \frac{8}{\sqrt{15}}$.

Exercise Group. Use the Cofunction Identities to determine the value of θ.

11. $\sin 28° = \cos\theta$

Answer. $\theta = 62°$

12. $\cos 74° = \sin\theta$

Answer. $\theta = 16°$

13. $\tan 52° = \cot\theta$

Answer. $\theta = 38°$

14. $\sec 87° = \csc\theta$

Answer. $\theta = 3°$

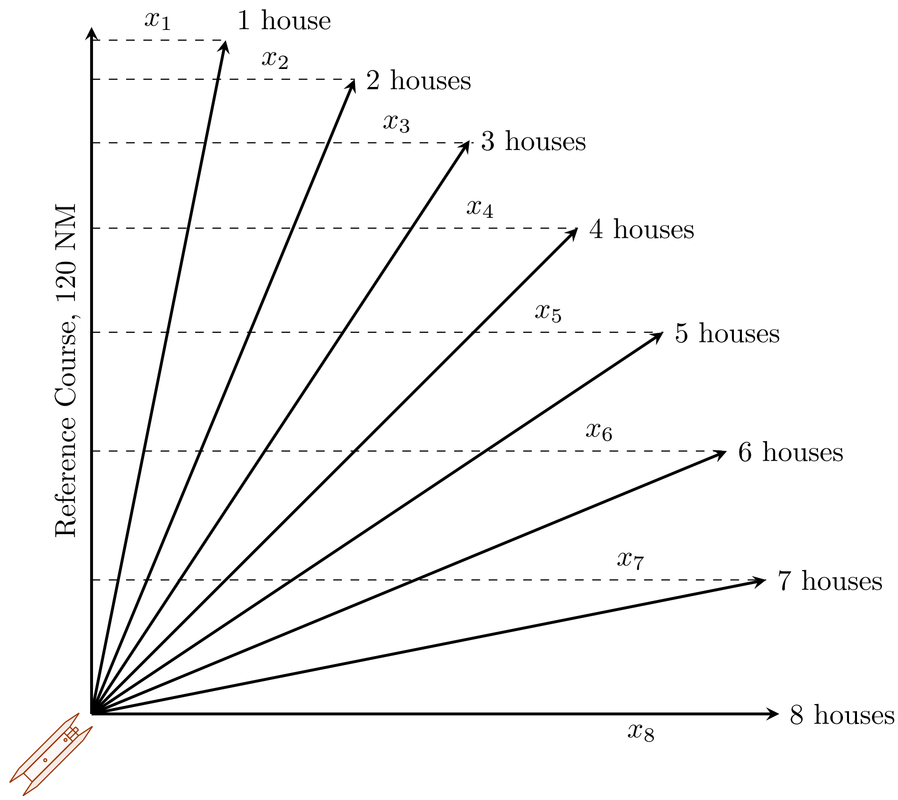
15. $\sin\frac{3\pi}{8} = \cos\theta$

Answer. $\theta = \frac{\pi}{8}$

16. $\cot\frac{2\pi}{5} = \tan\theta$

Answer. $\theta = \frac{\pi}{10}$

Exercise Group. In Example 1.4.15, we determined that when a wa‘a sails for one day (120 nautical miles) and deviates from its course by 1 house, the resulting deviation from the reference course is 23.4 NM. Now, calculate the deviations (x) for the remaining 7 angles. Round your answer to the nearest tenth of a nautical mile. Remember that one house corresponds to 11.25°.



17. 2 houses ($11.25° \times 2 = 22.5°$); x_2

Answer. $x_2 = 45.9$ NM

18. 3 houses ($11.25° \times 3 = 33.75°$); x_3

Answer. $x_3 = 66.7$ NM

19. 4 houses ($11.25° \times 4 = 45°$); x_4

Answer. $x_4 = 84.9$ NM

20. 5 houses ($11.25° \times 5 = 56.25°$); x_5

Answer. $x_5 = 99.8$ NM

21. 6 houses ($11.25° \times 6 = 67.5°$); x_6

Answer. $x_6 = 110.9$ NM

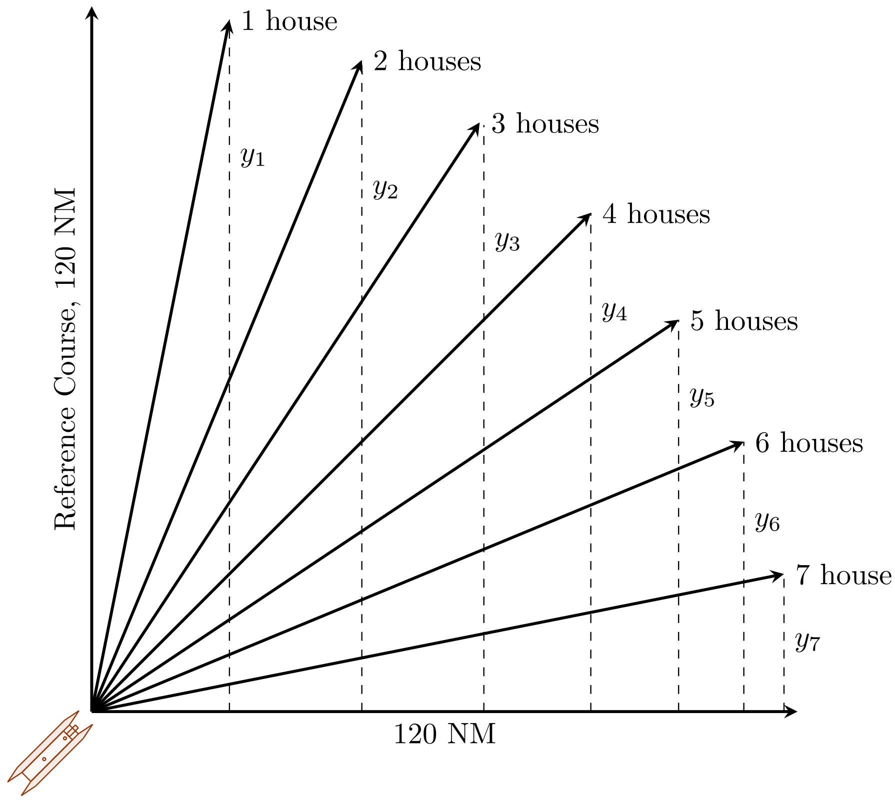
22. 7 houses ($11.25° \times 7 = 78.75°$); x_7

Answer. $x_7 = 117.7$ NM

23. 8 houses ($11.25° \times 8 = 90°$); x_8

Answer. $x_8 = 120$ NM

Exercise Group. In Exercise 1.4.7.17–23 we determined the deviation of a wa‘a following a day of sailing (120 nautical miles). Your task now is to calculate the distance the wa‘a has progressed along the reference course (north) for each deviation, denoted as y. Round your answer to the nearest tenth of a nautical mile and remember that one house corresponds to 11.25°.



24. 1 house ($11.25° \times 1 = 11.25°$); y_1

Answer. $y_1 = 117.7$ NM

25. 2 houses ($11.25° \times 2 = 22.5°$); y_2

Answer. $y_2 = 110.9$ NM

26. 3 houses ($11.25° \times 3 = 33.75°$); y_3

Answer. $y_3 = 99.8$ NM

27. 4 houses ($11.25° \times 4 = 45°$); y_4

Answer. $y_4 = 84.9$ NM

28. 5 houses ($11.25° \times 5 = 56.25°$); y_5

Answer. $y_5 = 66.7$ NM

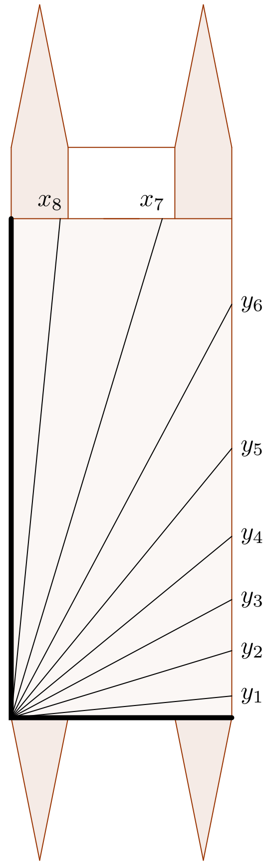
29. 6 houses ($11.25° \times 6 = 67.5°$); y_6

Answer. $y_6 = 45.9$ NM

30. 7 houses ($11.25° \times 7 = 78.75°$); y_7

Answer. $y_7 = 23.4$ NM

Exercise Group. One way to determine our bearing on a canoe is by observing and comparing the positions of celestial and other markers relative to our canoe. To facilitate this, we can mark the locations of the Star Compass on the opposite railings from the navigator's seat in the back corner of the canoe. However, since the Star Compass is circular and the canoe is rectangular, accurately placing the markings can be challenging.



When the navigator occupies the port stern (back left) corner of the deck, markers indicating the boundaries between houses can be placed on the corresponding railings on the bow (front) and starboard (right) sides of the canoe. For each value of θ, calculate the distance along the starboard railing (y) or bow railing (x) for a canoe with dimensions $l = 50$ ft and $w = 20$ ft. Round your answers to three decimal places.

31. $\theta = 5.625°$; y_1

Answer. $y_1 = 1.970$ ft

32. $\theta = 16.875°$; y_2

Answer. $y_2 = 6.067$ ft

33. $\theta = 28.125°$; y_3

Answer. $y_3 = 10.690$ ft

34. $\theta = 39.375°$; y_4

Answer. $y_4 = 16.414$ ft

35. $\theta = 50.625°$; y_5

Answer. $y_5 = 24.370$ ft

36. $\theta = 61.875°$; y_6

Answer. $y_6 = 37.417$ ft

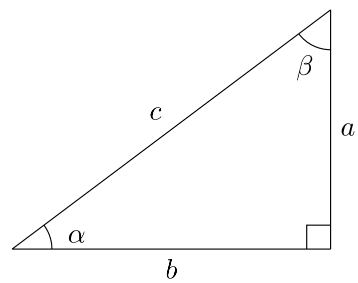
37. $\theta = 73.125°$; x_7

Answer. $x_7 = 15.167$ ft

38. $\theta = 84.375°$; x_8

Answer. $x_8 = 4.925$ ft

Exercise Group. Use the right triangle (not drawn to scale) provided below to solve for the given information. Round your solutions to two decimal places.



39. $a = 5$, $\beta = 35°$. Find b, c, and α

Answer. $b = 3.50$, $c = 6.10$, $\alpha = 55°$

40. $b = 12$, $\beta = 23°$. Find a, c, and α

Answer. $a = 28.27$, $c = 30.71$, $\alpha = 67°$

41. $b = 7$, $\alpha = 75°$. Find a, c, and β

Answer. $a = 26.12$, $c = 27.05$, $\beta = 15°$

42. $c = 4$, $\beta = 50°$. Find a, b, and α

Answer. $a = 2.57$, $b = 3.06$, $\alpha = 40°$

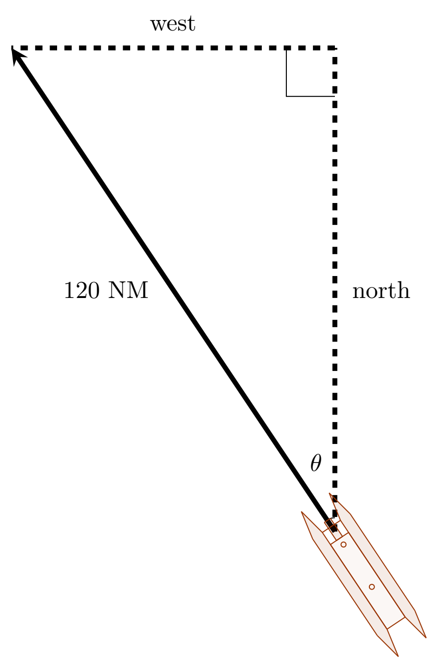
43. $c = 10$, $\alpha = 18°$. Find a, b, and β

Answer. $a = 3.09$, $b = 9.51$, $\beta = 72°$

44. $a = 6$, $\alpha = 38°$. Find b, c, and β

Answer. $b = 7.68$, $c = 9.75$, $\beta = 52°$

45. A wa‘a sails in the direction of the house Nālani Ho‘olua for one day, covering 120 nautical miles. How many nautical miles has the wa‘a traveled north? How many miles has the wa‘a traveled west? To calculate the angle θ, refer to the Star Compass (Figure 1.1.4) to determine the number of houses, and use the fact that one house is 11.25°.

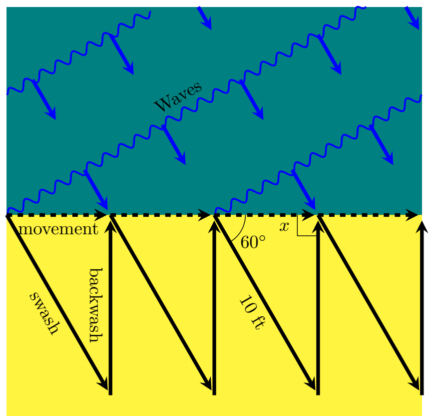


Answer. 66.7 NM west; 99.8 NM north

46. **Movement of Sand.** The movement of sand on a beach is a dynamic process influenced by various factors, such as waves. When waves approach the shore at an angle, they lead to the shifting of sand. During the *swash* phase, as the wave crashes onto the shore, water and sediment move onto the beach following the wave's angle. Subsequently, gravity propels the water and sediment back into the ocean, perpendicular to the shoreline, in a process known as *backwash*. This interplay of swash and backwash creates a zig-zag pattern called *longshore drift*.

Certain beaches undergo seasonal changes in wave direction. Some experience waves from one direction in one season and from another direction in the next, while those receiving waves predominantly from a single direction might accumulate sand in specific areas.

Calculate how far along the shore a single grain of sand moves after a wave breaks at a 60° angle and travels onto the shore for 10 ft before receding back into the ocean.



Answer. 5ft

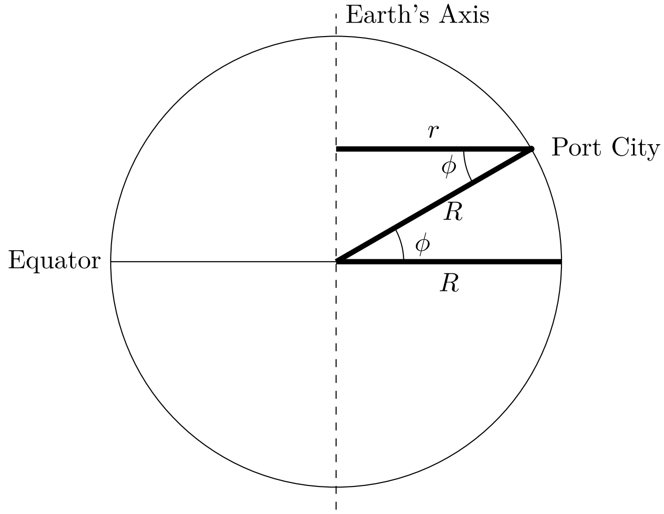
Exercise Group. Between 2013 and 2017, Hōkūleʻa completed a global circumnavigation with a mission *mālama honua* - "care for our Earth" and to foster a sense of *ʻohana* ("family") for all people and places. This remarkable voyage spanned 40,000 nautical miles and made stops at over 150 ports across 18 nations.

Throughout this voyage, Earth's rotation occurs around an axis that extends from the North Pole to the South Pole. The rotation imparts an angular speed and linear velocity to every point on Earth. Assuming Earth completes one rotation within 24 hours and treating Earth as a perfect sphere with a radius of $R = 4,000$ miles, we can calculate the following parameters for each of the Mālama Honua Voyage's ports, given their latitudes (ϕ).

(a) Calculate r, the distance from the port to Earth's axis of rotation (in miles, rounded to one decimal place).

(b) Calculate ω, the angular velocity (in radians per hour, rounded to four decimal places).

(c) Calculate v, the linear speed (in miles per hour, rounded to the nearest whole number).



47. Hilo, Hawai‘i (19.7216° N)

Answer.

(a) $r = 3,765.4$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 986$ mi/hr

48. Papeete, Tahiti (17.5325° S)

Answer.

(a) $r = 3,814.2$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 999$ mi/hr

49. Apia, Samoa (13.8507° S)

Answer.

(a) $r = 3,883.7$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,017$ mi/hr

50. Waitangi, Aotearoa (35.2683° S)

Answer.

(a) $r = 3,265.8$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 855$ mi/hr

51. Sydney, Australia (33.8688° S)

Answer.

(a) $r = 3,321.3$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 870$ mi/hr

52. Bali, Indonesia (8.4095° S)

Answer.

(a) $r = 3,957.0$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,036$ mi/hr

53. Port Louis, Mauritius (20.1609° S)

Answer.

(a) $r = 3,754.9$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 983$ mi/hr

54. Cape Town, South Africa (33.9249° S)

Answer.

(a) $r = 3,319.1$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 869$ mi/hr

55. Natal, Brazil (5.7842° S)
Answer.

(a) $r = 3,979.6$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,042$ mi/hr

56. Necker, British Virgin Islands (18.5268° N)
Answer.

(a) $r = 3,792.7$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 993$ mi/hr

57. Yarmouth, Nova Scotia (43.8379° N)
Answer.

(a) $r = 2,885.2$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 755$ mi/hr

58. Balboa, Panama (8.9614°N)
Answer.

(a) $r = 3,951.2$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,034$ mi/hr

59. Galapagos Islands (0.9538° S)
Answer.

(a) $r = 3,999.5$ miles;

(b) $\omega = 0.2618$ rad/hr;

(c) $v = 1,047$ mi/hr

60. Rapa Nui (27.1127° S)
Answer.

(a) $r = 3,560.5$ miles;

(b) $\omega = 0.2618$ rad/hr;

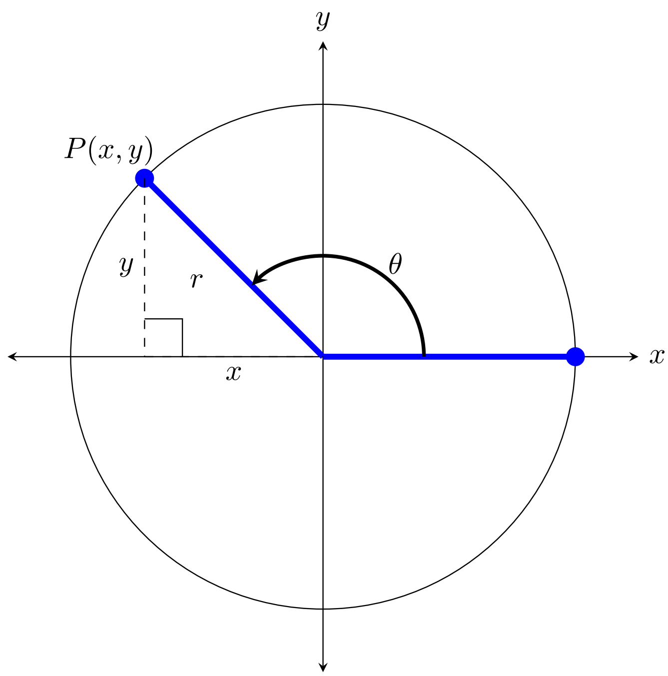
(c) $v = 932$ mi/hr

1.5 Trigonometric Functions of Any Angle

Now that we have been introduced to the six trigonometric functions for special angles in the first quadrant, we can explore their properties across all quadrants.

1.5.1 Determine the Signs of the Trigonometric Functions Based on the Quadrant

Let $P(x, y)$ be a point on the circle. The signs of the six trigonometric functions vary depending on the quadrant in which $P(x, y)$ lies in.



Example 1.5.1 Let $P(x, y)$ be in Quadrant II. Determine the sign for each of the six trigonometric functions.

Solution. Since $P(x, y)$ is in Quadrant II, $x < 0$ and $y > 0$. Note that $r > 0$. Then we have

$$\sin\theta = \frac{y}{r} = \frac{(+)}{(+)} = (+), \quad \cos\theta = \frac{x}{r} = \frac{(-)}{(+)} = (-), \quad \tan\theta = \frac{y}{x} = \frac{(+)}{(-)} = (-),$$
$$\csc\theta = \frac{r}{y} = \frac{(+)}{(+)} = (+), \quad \sec\theta = \frac{r}{x} = \frac{(+)}{(-)} = (-), \quad \cot\theta = \frac{x}{y} = \frac{(-)}{(+)} = (-).$$

□

We can check the remaining quadrants using a similar approach. Table 1.5.2 and Figure 1.5.3 provide a list of the signs of the six trigonometric functions for each quadrant.

Table 1.5.2 Signs of the trigonometric functions.

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sinθ, cscθ	cosθ, secθ, tanθ, cotθ
III	tanθ, cotθ	sinθ, cscθ, cosθ, secθ
IV	cosθ, secθ	sinθ, cscθ, tanθ, cotθ

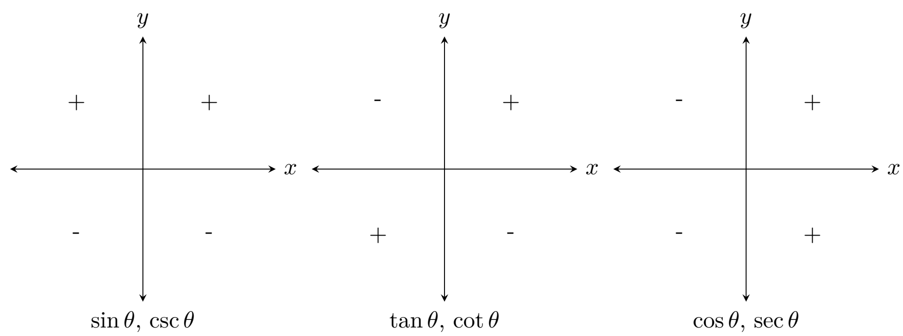


Figure 1.5.3 Signs of trigonometric functions.

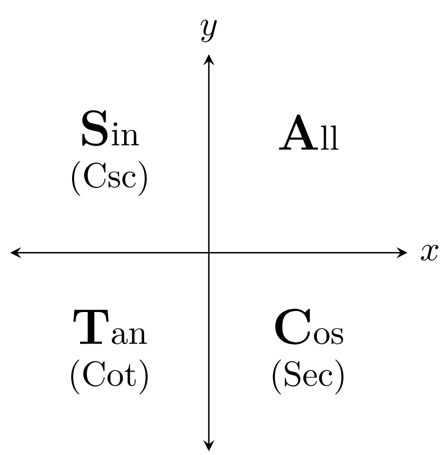
Example 1.5.4 If $\sin\theta < 0$ and $\cos\theta > 0$, what quadrant does θ lie in?

Solution. Since $\sin\theta < 0$, then θ is either in Quadrant III or IV. However, we also have $\cos\theta > 0$ which means that θ is either in Quadrant I or IV. Thus the only quadrant that satisfies both conditions is Quadrant IV. □

Mnemonic devices for remembering the quadrants in which the trigonometric functions are positive are

- "*A* *S*mart *T*rig *C*lass"
- "*A*ll *S*tudents *T*ake *C*alculus"

which correspond to "*A*ll *S*in *T*an *C*os."



Example 1.5.5 Let $\sin\theta = -\frac{12}{13}$ and $\cos\theta = -\frac{5}{13}$. Compute the exact values of the remaining trigonometric functions of θ using identities.

Solution. Since $\sin\theta < 0$ and $\cos\theta < 0$, we refer to Table 1.5.2 and see that θ is in Quadrant III. From Table 1.5.2 we know $\tan\theta > 0$, $\csc\theta < 0$, $\sec\theta < 0$,

$\cot\theta > 0$. From the Quotient Identity (Definition 1.4.3), we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}.$$

Next, using the Reciprocal Identities (Definition 1.4.2), we get

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{12}{13}} = -\frac{13}{12},$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5},$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{12}{5}} = \frac{5}{12}.$$

□

1.5.2 Reference Angles

Now that we can determine the signs of trigonometric functions, we will demonstrate how the value of any trigonometric function at any angle can be found from its value in Quadrant I (between 0° and 90° or 0 and $\frac{\pi}{2}$).

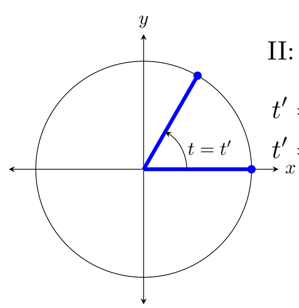
Definition 1.5.6 Let t be a real number. A **reference angle**, t', is the acute angle ($< 90°$) formed by the terminal side of angle t and the x-axis. In other words, it is the shortest arc length along the unit circle measured between the terminal side and the x-axis. Angles in Quadrant I are their own reference angles. ♢

Remark 1.5.7 Calculating the reference angle. To calculate the reference angle t' for a given angle t:

- In radians, if $t > 2\pi$ or if $t < 0$, add or subtract multiples of 2π to obtain a coterminal angle between 0 and 2π. Then, find the reference angle.
- In degrees, if $t > 360°$ or $t < 0°$, add or subtract multiples of 360° to obtain a coterminal angle between 0° and 360°. Then, find the reference angle.

Quadrant I:

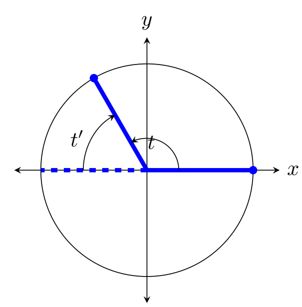
$t' = t$



Quadrant II:

$t' = \pi - t$

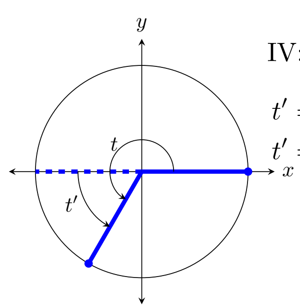
$t' = 180° - t$



Quadrant III:

$t' = t - \pi$

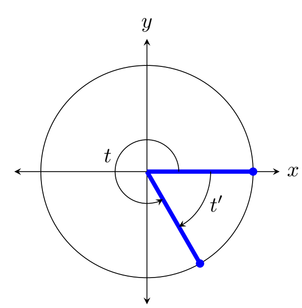
$t' = t - 180°$



Quadrant IV:

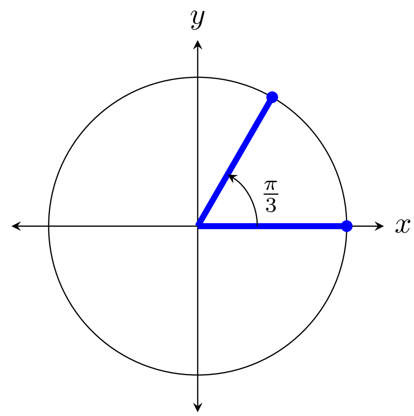
$t' = 2\pi - t$

$t' = 360° - t$



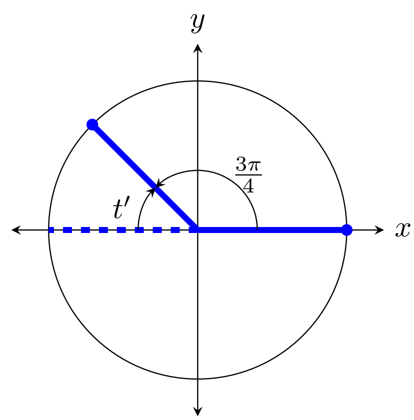
Example 1.5.8 Find the reference angle for each value of t.

(a) $t = \frac{\pi}{3}$



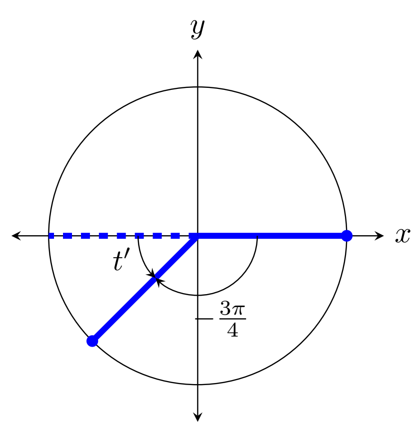
Solution. The angle $t = \frac{\pi}{3}$ is in the first quadrant and so it is its own reference angle: $t = t' = \frac{\pi}{3}$.

(b) $t = \frac{3\pi}{4}$



Solution. From the figure, we see that the shortest arc length to the x-axis is in the direction of π. We see that $t' + \frac{3\pi}{4} = \pi$ so $t' = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

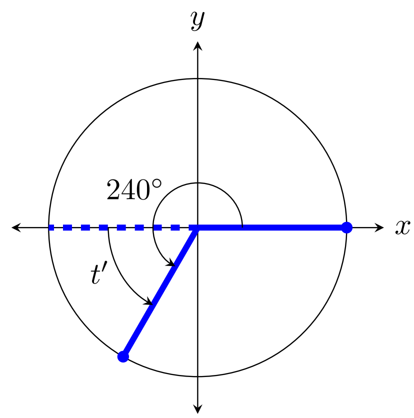
(c) $t = -\frac{3\pi}{4}$



Solution 1. Since $t < 0$, we can add 2π to get $-\frac{3\pi}{4} + 2\pi = \frac{5\pi}{4}$. From the formula, we get $t' = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

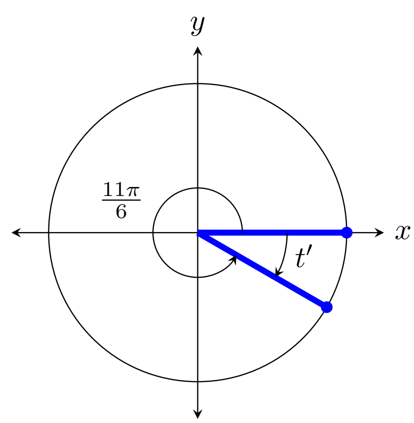
Solution 2. Since $-\frac{3\pi}{4}$ spans only two quadrants counterclockwise, we can treat it similarly to an angle in Quadrant II. By the previous problem, $t' + \frac{3\pi}{4} = \pi$ so $t' = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

(d) $t = 240°$



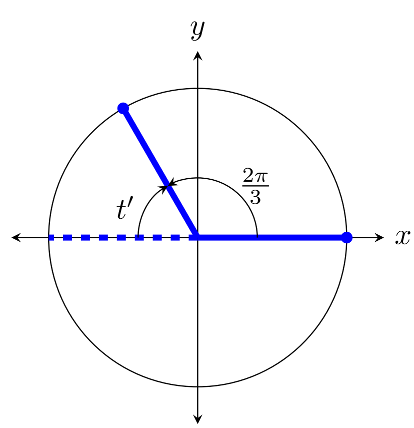
Solution. From the figure we see that the shortest arc length to the x-axis is towards 180°. We observe that $240° - t' = 180°$ so $t' = 240° - 180° = 60°$.

(e) $t = \frac{11\pi}{6}$



Solution. From the figure, we see that the shortest arc length to the x-axis is towards 2π. We observe that $t' + \frac{11\pi}{6} = 2\pi$ so $t' = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$.

(f) $t = \frac{2\pi}{3}$



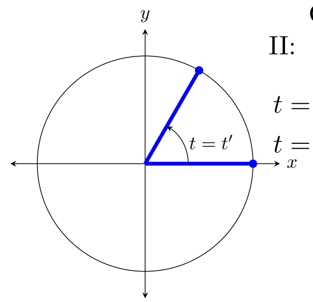
Solution. From the figure, we see that the shortest arc length to the x-axis is towards π. We observe that $t' + \frac{2\pi}{3} = \pi$ so $t' = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

□

Remark 1.5.9 Calculate an angle in standard position given its quadrant and reference angle. We calculate an angle in standard position, t, given the quadrant that t lies in and the reference angle t'.

Quadrant I:

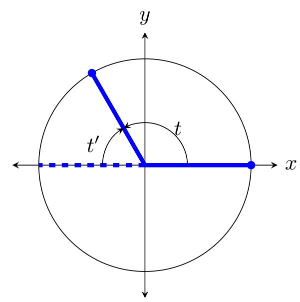
$t = t'$



Quadrant II:

$t = \pi - t'$

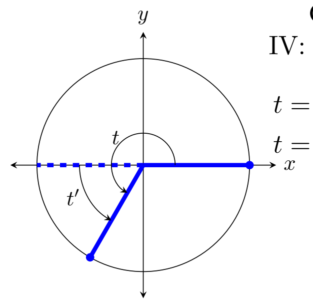
$t = 180° - t'$



Quadrant III:

$t = t' - \pi$

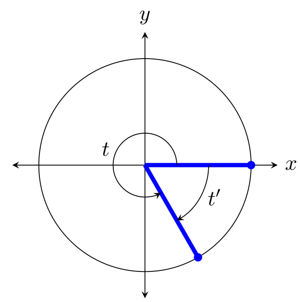
$t = t' - 180°$



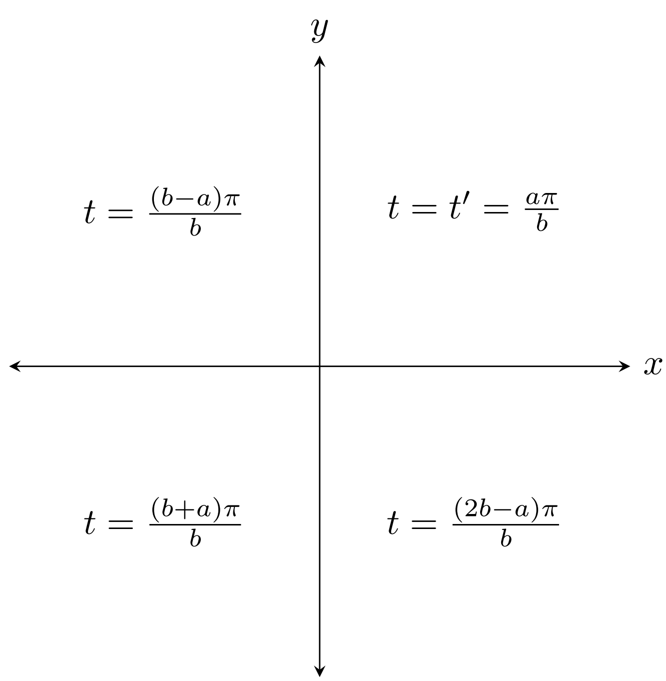
Quadrant IV:

$t = 2\pi - t'$

$t = 360° - t'$

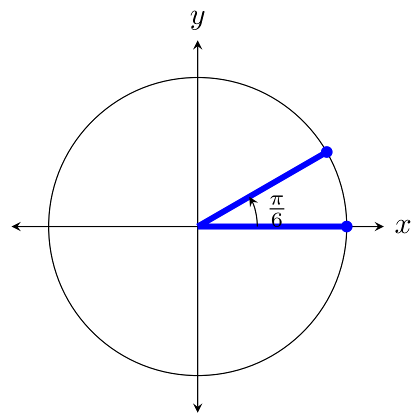


For radians only: If the reference angle (in radians) is of the form $t' = \frac{a\pi}{b}$, then the associated angle in standard position, t, can be calculated by:



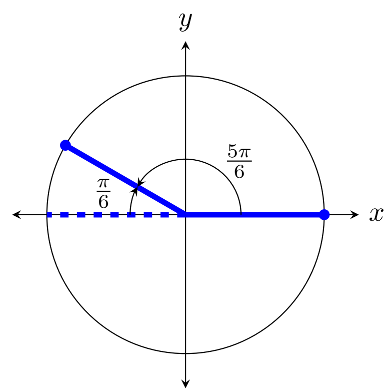
Example 1.5.10 Calculate an angle given its reference angle and quadrant. Given a reference angle, t, compute the associated angle in standard position for Quadrants II, III, and IV.

(a) $t' = \frac{\pi}{6}$



(i) Quadrant II

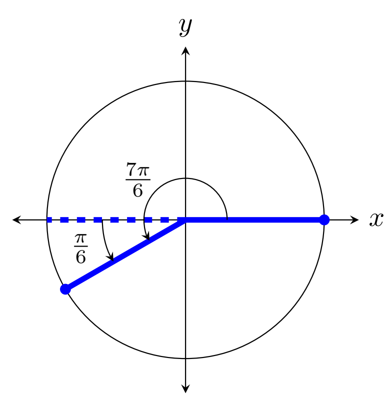
Solution 1. In Quadrant II, the associated angle is $t = \pi - \frac{\pi}{6} = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}$.



Solution 2. Since $t' = \frac{\pi}{6} = \frac{1\pi}{6}$, then $t' = \frac{(6-1)\pi}{6} = \frac{5\pi}{6}$.

(ii) Quadrant III

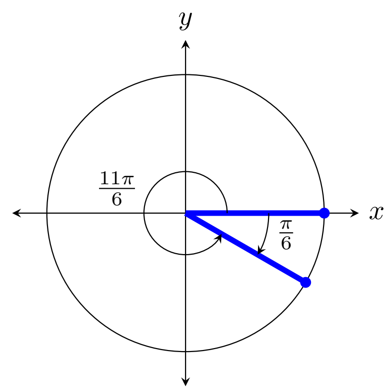
Solution 1. In Quadrant III, the associated angle is $t = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}$.



Solution 2. $t' = \frac{(6+1)\pi}{6} = \frac{7\pi}{6}$.

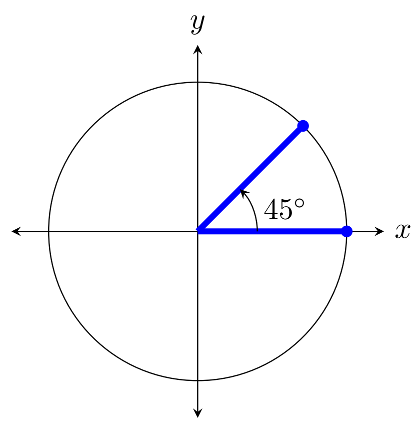
(iii) Quadrant IV

Solution 1. In Quadrant IV, the associated angle is $t = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$.



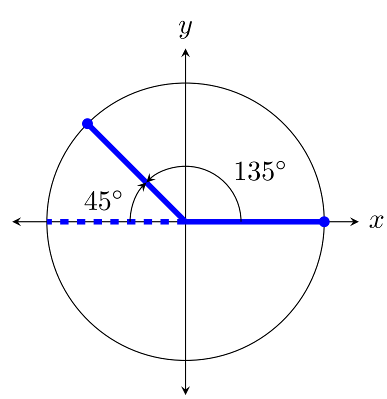
Solution 2. $t' = \frac{(2\cdot 6-1)\pi}{6} = \frac{11\pi}{6}$.

(b) $t' = 45°$



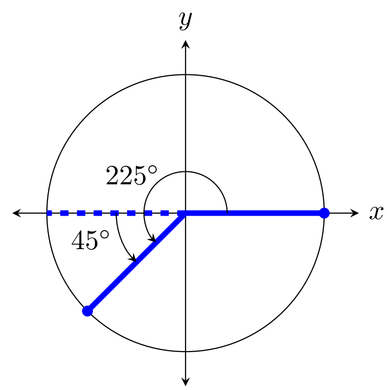
(i) Quadrant II

Solution. In Quadrant II, the associated angle is $t = 180° - 45° = 135°$.



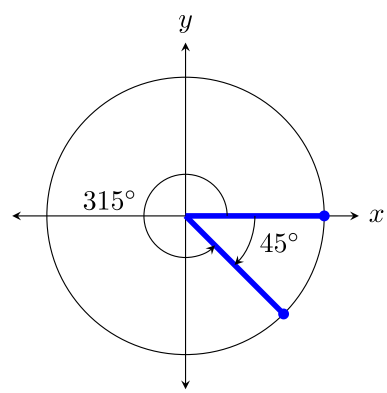
(ii) Quadrant III

Solution. In Quadrant III, the associated angle is $t = 180° + 45° = 225°$.



(iii) Quadrant IV

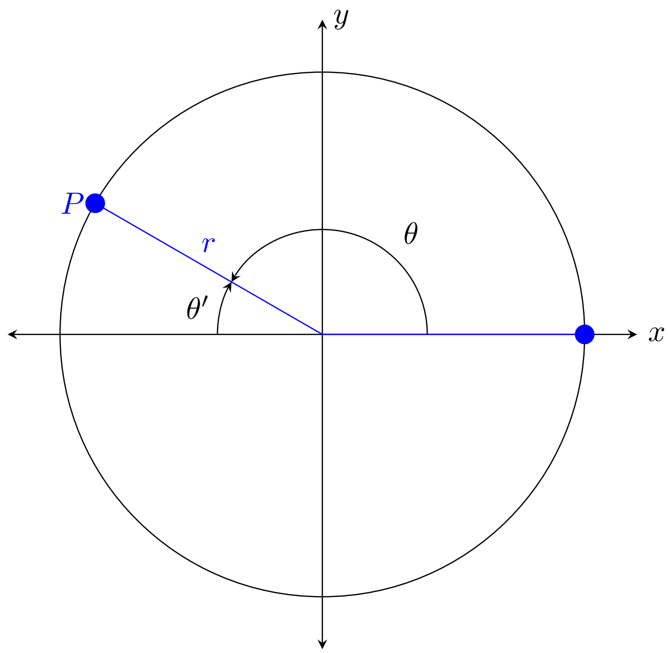
Solution. In Quadrant IV, the associated angle is $t = 360° - 45° = 315°$.



□

1.5.3 Evaluating Trigonometric Functions Using Reference Angles

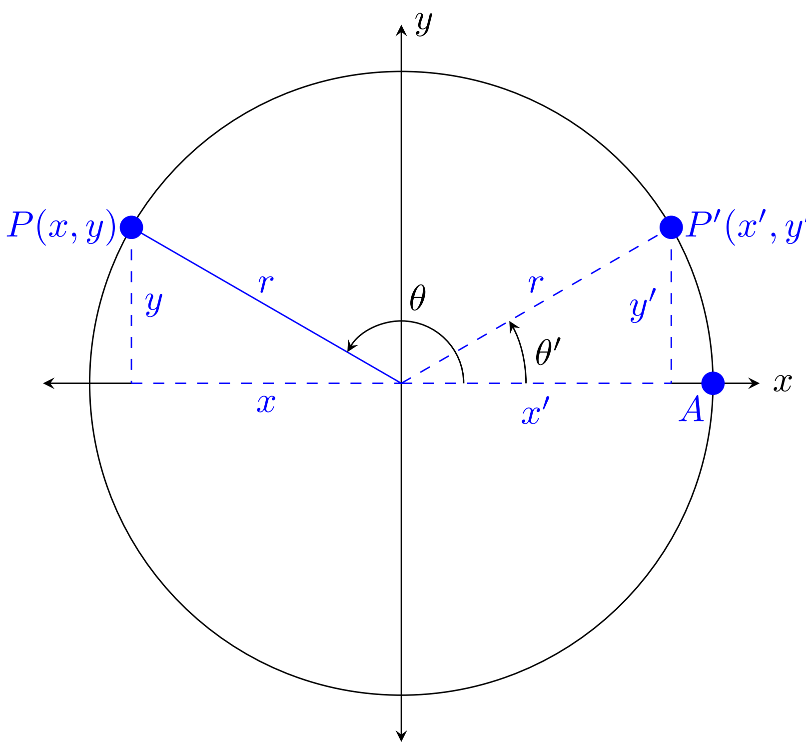
To evaluate trigonometric functions in any quadrant using reference angles, we begin with an angle, θ, that lies in Quadrant II. When evaluating $\sin\theta$ and $\cos\theta$, we begin by plotting θ in standard position and then proceed to determine and draw its corresponding reference angle, θ'.



By definition we know that

$$\sin\theta = \frac{y}{r}; \qquad \cos\theta = \frac{x}{r}.$$

Next, we draw the reference angle, θ', in standard position.



We now have

$$\sin\theta' = \frac{y'}{r}; \qquad \cos\theta' = \frac{x'}{r}.$$

Notice that the y-coordinates for P and P' share the same value, thus $y = y'$

and we get

$$\sin\theta = \sin\theta'.$$

Similarly, we can see that the x-coordinates of P and P' have opposite values, thus $x = -x'$ and

$$\cos\theta = -\cos\theta'.$$

You may have noticed that we have two similar triangles, differing only in their x-coordinates have opposite values. Consequently, the values of each trigonometric function for the two triangles will match, except for a potential difference in signs. The sign of each function can be deduced by referring to Table 1.5.2. This approach is applicable across all quadrants. To sum up, we now outline the steps for utilizing reference angles to evaluate trigonometric functions.

Remark 1.5.11 Steps for Evaluating Trigonometric Functions Using Reference Angles. The values of a trigonometric function for a specific angle are equivalent to the values of the same trigonometric function for the reference angle, with a potential difference in sign. To compute the value of a trigonometric function for any angle, use the following steps

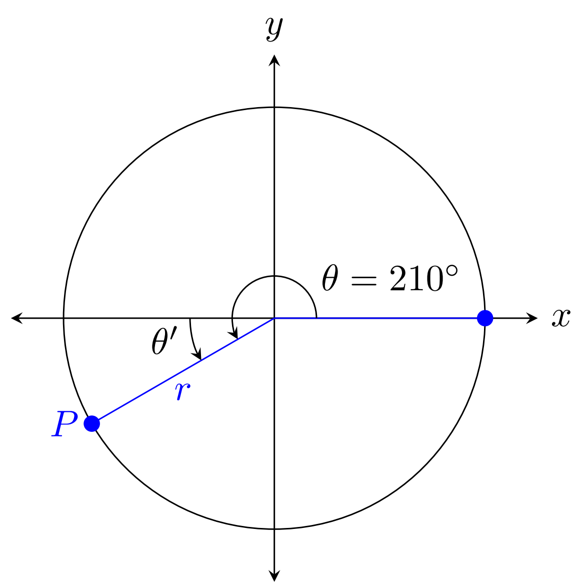
1. Draw the angle in standard position.
2. Determine the reference angle associated with the angle.
3. Evaluate the trigonometric function at the reference angle.
4. Use Table 1.5.2 and the quadrant of the original angle to determine the appropriate sign for the function.

Example 1.5.12 Use the reference angle associated with the given angle to find the exact value of:

(a) $\cos 210°$

Solution. We will use the steps for evaluating trigonometric functions using reference angles.

(a) First we draw the angle $\theta = 210°$.



(b) The reference angle is

$$\theta' = 210^\circ - 180^\circ = 30^\circ.$$

(c) $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

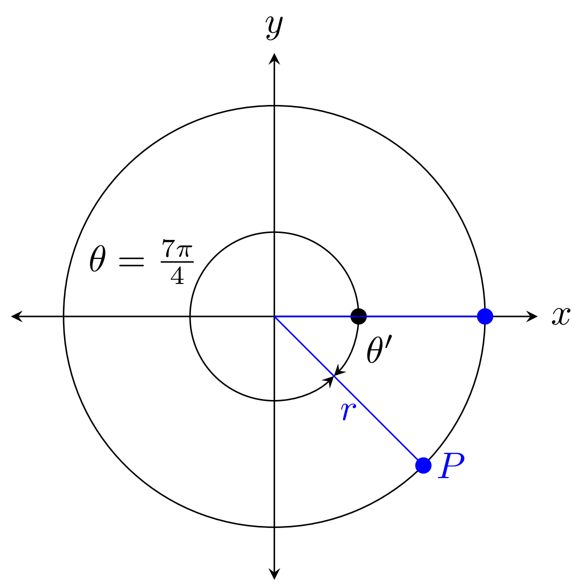
(d) Since 210° lies in Quadrant III, we know that $\cos\theta < 0$, so

$$\cos 210^\circ = -\frac{\sqrt{3}}{2}.$$

(b) $\tan \frac{7\pi}{4}$

Solution. We will use the steps for evaluating trigonometric functions using reference angles.

(a) First we draw the angle $\theta = \frac{7\pi}{4}$.



(b) The reference angle is

$$2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}.$$

(c) $\tan \frac{\pi}{4} = 1$.

(d) Since $\frac{7\pi}{4}$ lies in Quadrant IV, we know that $\tan\theta < 0$, so

$$\tan \frac{7\pi}{4} = -1.$$

□

Example 1.5.13 Calculate $\sin\theta$ and $\cos\theta$ if $\theta = \frac{20\pi}{3}$.
Solution.

1. First we draw the angle $\theta = \frac{20\pi}{3}$.

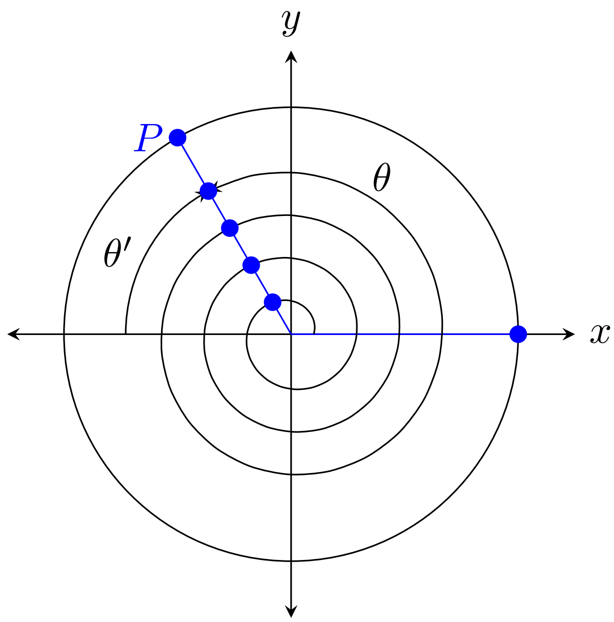


Figure 1.5.14 The angle $\theta = \frac{20\pi}{3}$ makes three rotations before ending in Quadrant II.

2. To obtain the reference angle, we first subtract multiples of 2π from θ to obtain a coterminal angle between 0 and 2π:

$$\frac{20\pi}{3} - 2\pi = \frac{20\pi}{3} - \frac{6\pi}{3} = \frac{14\pi}{3},$$
$$\frac{20\pi}{3} - 4\pi = \frac{20\pi}{3} - \frac{12\pi}{3} = \frac{8\pi}{3},$$
$$\frac{20\pi}{3} - 6\pi = \frac{20\pi}{3} - \frac{18\pi}{3} = \frac{2\pi}{3}.$$

From Example 1.5.8, the reference angle for $\frac{2\pi}{3}$ is $\theta' = \frac{\pi}{3}$.

3. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$.

4. Since $\frac{20\pi}{3}$ lies in Quadrant II, we know that $\sin\theta > 0$ and $\cos\theta < 0$, so

$$\sin\frac{20\pi}{3} = \frac{\sqrt{3}}{2}; \qquad \cos\frac{20\pi}{3} = -\frac{1}{2}.$$

□

1.5.4 Periodic Functions

In Figure 1.5.14 of Example 1.5.13, point P corresponds to the angle $\frac{20\pi}{3}$. To determine the reference angle, we subtracted multiples of 2π. Each iteration of 2π retraces the unit circle back to the point P, resulting in a coterminal angle. Therefore

$$\sin\frac{2\pi}{3} = \sin\frac{8\pi}{3} = \sin\frac{14\pi}{3} = \sin\frac{20\pi}{3}.$$

Rewriting the angles we get

$$\sin\left(\frac{2\pi}{3} + 0\cdot 2\pi\right) = \sin\left(\frac{2\pi}{3} + 1\cdot 2\pi\right) = \sin\left(\frac{2\pi}{3} + 2\cdot 2\pi\right) = \sin\left(\frac{2\pi}{3} + 3\cdot 2\pi\right).$$

Similarly,

$$\cos\left(\frac{2\pi}{3} + 0 \cdot 2\pi\right) = \cos\left(\frac{2\pi}{3} + 1 \cdot 2\pi\right) = \cos\left(\frac{2\pi}{3} + 2 \cdot 2\pi\right) = \cos\left(\frac{2\pi}{3} + 3 \cdot 2\pi\right).$$

In general, consider an angle θ measured in radians and its corresponding point P on the unit circle. Adding or subtracting integer multiples of 2π to θ will lead to the same point P on the unit circle. Thus, the values of sine and cosine for all angles corresponding to point P are equivalent. This leads us to the following periodic properties.

Definition 1.5.15 Periodic Properties.

$$\sin(\theta + 2\pi k) = \sin\theta \qquad \cos(\theta + 2\pi k) = \cos\theta$$

where k is any integer. ◊

Functions such as these that repeat their values in regular cycles are called *periodic functions*.

Definition 1.5.16 A function f is called **periodic** if there exists a positive number p such that

$$f(\theta + p) = f(\theta)$$

for every θ. The smallest number p is called the **period** of f. ◊

Sine, cosine, cosecant, and secant repeat their values with a period of 2π while tangent and cotangent have a period of π.

Definition 1.5.17 Periodic Properties.

$$\begin{array}{lll} \sin(\theta + 2\pi) = \sin\theta, & \cos(\theta + 2\pi) = \cos\theta, & \tan(\theta + \pi) = \tan\theta, \\ \csc(\theta + 2\pi) = \csc\theta, & \sec(\theta + 2\pi) = \sec\theta, & \cot(\theta + \pi) = \cot\theta. \end{array}$$

◊

1.5.5 Trigonometric Table

The Trigonometric Identities and reference angles give us the values of trigonometric functions in Table 1.5.18.

Table 1.5.18 Values of the six trigonometric functions for common angles (Undefined values are abbreviated as "undef").

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0°$	0	0	1	0	undef	1	undef
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$90°$	$\frac{\pi}{2}$	1	0	undef	1	undef	0
$120°$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$
$135°$	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$150°$	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
$180°$	π	0	-1	0	undef	-1	undef
$210°$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$225°$	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$240°$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2	$\frac{1}{\sqrt{3}}$
$270°$	$\frac{3\pi}{2}$	-1	0	undef	-1	undef	0
$300°$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$
$315°$	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$330°$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	$-\sqrt{3}$

Remark 1.5.19 Table Made Easy. Table 1.5.18 may seem intimidating but if we recognize the symmetry about 90°, 180°, and 270°, we will only need to focus on the values for the first quadrant (Table 1.4.6). In fact, we need only produce the values of sine in Quadrant I. Use the Cofunction Identities (Definition 1.4.7) to find the values of cosine. Next, apply the trigonometric identity to find $\tan\theta = \sin\theta/\cos\theta$. Finally, use the the Reciprocal Identities (Definition 1.4.2) to produce $\csc\theta$, $\sec\theta$, and $\cot\theta$.

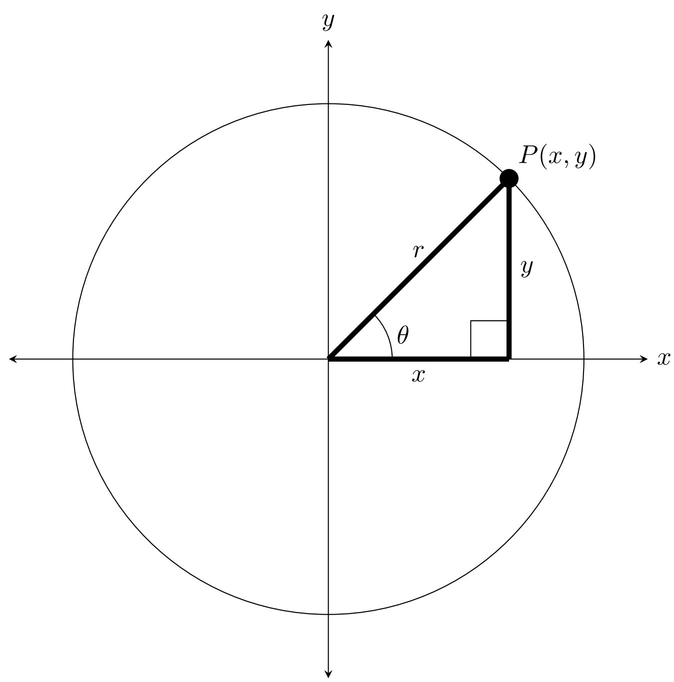
1.5.6 Pythagorean Identities

Definition 1.5.20 Pythagorean Identities.

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

◊

Proof. We will use the Pythagorean Theorem to prove the reciprocal identities.



Let $P(x, y)$ be a point on the circle with radius r. The equation of the circle is

$$x^2 + y^2 = r^2.$$

By definition $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$. Thus we have

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

which is our first Pythagorean Identity. The proofs of the remaining identities are left as exercises. ■

Example 1.5.21 Let θ be an angle in Quadrant IV and let $\cos\theta = \frac{3}{5}$. Calculate the exact values of $\sin\theta$ and $\tan\theta$.

Solution. Substituting our value of $\cos\theta$ into the Pythagorean identity, we obtain:

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1\\
\sin^2\theta + \left(\frac{3}{5}\right)^2 &= 1\\
\sin^2\theta + \frac{9}{25} &= 1\\
\sin^2\theta &= 1 - \frac{9}{25}\\
\sin^2\theta &= \frac{16}{25}.
\end{aligned}$$

Taking the square root of both sides,

$$\sin\theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}.$$

Since θ is in Quadrant II, we have $\sin\theta < 0$. Thus we choose the negative answer to get

$$\sin\theta = -\frac{4}{5}.$$

Next we use the Quotient Identity to get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{5}\cdot\frac{5}{3} = -\frac{4}{3}.$$

□

1.5.7 Even and Odd Trigonometric Functions

Recall that a function f is even if $f(-x) = f(x)$ for all values of x, and a function is odd if $f(-x) = -f(x)$ for all values of x. With this understanding, we can now classify trigonometric functions as either even or odd.

Definition 1.5.22 Even and Odd Trigonometric Properties. The cosine and secant functions are **even**.

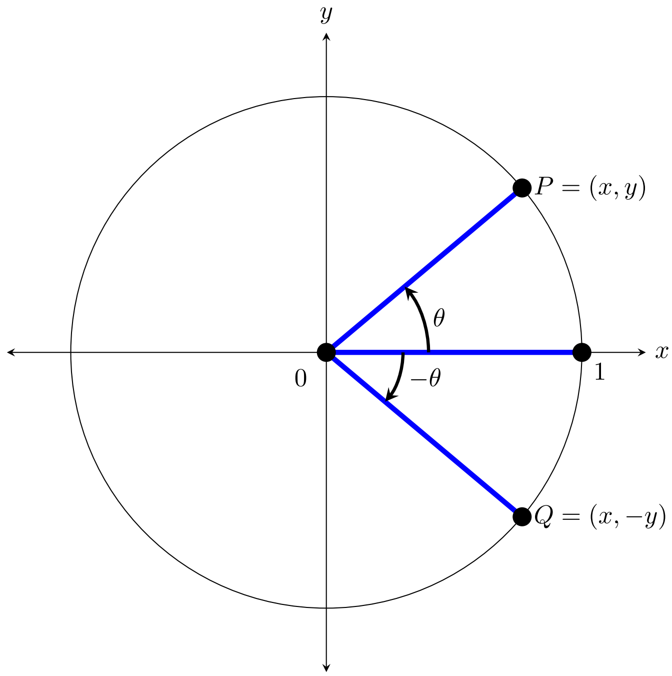
$$\cos(-\theta) = \cos\theta, \qquad \sec(-\theta) = \sec\theta.$$

The sine, cosecant, tangent, and cotangent functions are **odd**.

$$\begin{aligned}
\sin(-\theta) &= -\sin\theta, & \csc(-\theta) &= -\csc(\theta),\\
\tan(-\theta) &= -\tan\theta, & \cot(-\theta) &= -\cot(\theta).
\end{aligned}$$

◊

Proof. Let P be a point on the unit circle corresponding to the angle θ with coordinates (x, y) and Q be the point corresponding to the angle $-\theta$ with coordinates $(x, -y)$.



Using the Definition 1.3.4 for the six trigonometric functions we have

$$\sin\theta = y, \qquad \sin(-\theta) = -y, \qquad \cos\theta = x, \qquad \cos(-\theta) = x.$$

So

$$\sin(-\theta) = -y = -\sin\theta, \qquad \cos(-\theta) = x = \cos\theta.$$

Thus we conclude that sine is an odd function and cosine is an even function. Next, using the Quotient (Definition 1.4.3) and Reciprocal Identities (Definition 1.4.2) we get

$$\begin{aligned}
\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta,\\
\cot(-\theta) &= \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta,\\
\csc(-\theta) &= \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta,\\
\sec(-\theta) &= \frac{1}{\cos(-\theta)} = \frac{1}{-\cos\theta} = \sec\theta.
\end{aligned}$$

Thus tangent, cotangent, cosecant are odd functions and secant is an even function. ■

Example 1.5.23 Use the even-odd properties of trigonometric functions to determine the exact value of:

(a) $\csc(-30°)$

Solution. Since cosecant is an odd function, the cosecant of a negative angle has the opposite sign of the cosecant of the positive angle. Thus, $\csc(-30°) = -\csc 30° = -2$.

(b) $\cos(-\theta)$ if $\cos\theta = 0.4$

Solution. Cosine is an even function so $\cos(-\theta) = \cos\theta = 0.4$.

□

1.5.8 Exercises

Exercise Group. Determine the quadrant containing θ given the following:

1. $\cot\theta < 0$ and $\cos\theta < 0$

Answer. QII

2. $\csc\theta > 0$ and $\tan\theta > 0$

Answer. QI

3. $\cos\theta > 0$ and $\sin\theta < 0$

Answer. QIV

4. $\sec\theta > 0$ and $\tan\theta > 0$

Answer. QI

5. $\tan\theta < 0$ and $\csc\theta > 0$

Answer. QII

6. $\cot\theta > 0$ and $\sin\theta < 0$

Answer. QIV

7. $\sec\theta < 0$ and $\csc\theta < 0$

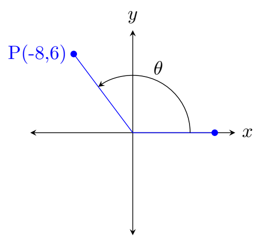
Answer. QIII

8. $\cos\theta < 0$ and $\tan\theta > 0$

Answer. QIII

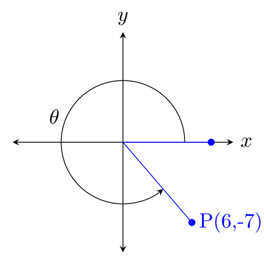
Exercise Group. The point $P(x, y)$ is on the terminal side of angle θ. Determine the exact values of the six trigonometric functions at θ.

9.



Answer. $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

10.



Answer. $\sin\theta = -\frac{7}{\sqrt{85}}$, $\cos\theta = \frac{6}{\sqrt{85}}$, $\tan\theta = -\frac{7}{6}$, $\csc\theta = -\frac{\sqrt{85}}{7}$, $\sec\theta = \frac{\sqrt{85}}{6}$, $\cot\theta = -\frac{6}{7}$

11. $(3, -4)$

Answer. $\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

12. $(-12, -5)$

Answer. $\sin\theta = -\frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\csc\theta = -\frac{13}{5}$, $\sec\theta = -\frac{13}{12}$, $\cot\theta = \frac{12}{5}$

13. $(-2, -3)$

Answer. $\sin\theta = -\frac{3}{\sqrt{13}}$, $\cos\theta = -\frac{2}{\sqrt{13}}$, $\tan\theta = \frac{3}{2}$, $\csc\theta = -\frac{\sqrt{13}}{3}$, $\sec\theta = -\frac{\sqrt{13}}{2}$, $\cot\theta = \frac{2}{3}$

14. $(-4, 4)$

Answer. $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\tan\theta = -1$ $\csc\theta = \sqrt{2}$, $\sec\theta = -\sqrt{2}$, $\cot\theta = -1$

15. $(-24, 7)$

Answer. $\sin\theta = \frac{7}{25}$, $\cos\theta = -\frac{24}{25}$, $\tan\theta = -\frac{7}{24}$, $\csc\theta = \frac{25}{7}$, $\sec\theta = -\frac{25}{24}$, $\cot\theta = -\frac{24}{7}$

16. $(9, -40)$

Answer. $\sin\theta = -\frac{40}{41}$, $\cos\theta = \frac{9}{41}$, $\tan\theta = -\frac{40}{9}$, $\csc\theta = -\frac{41}{40}$, $\sec\theta = \frac{41}{9}$, $\cot\theta = -\frac{9}{40}$

Exercise Group. Find the exact value of the remaining five trigonometric functions of θ from the given information.

17. $\tan\theta = -\frac{12}{5}$, θ is Quadrant II

Answer. $\sin\theta = \frac{12}{13}$, $\cos\theta = -\frac{5}{13}$, $\csc\theta = \frac{13}{12}$, $\sec\theta = -\frac{13}{5}$, $\cot\theta = -\frac{5}{12}$

18. $\cos\theta = \frac{3}{5}$, θ is Quadrant IV

Answer. $\sin\theta = -\frac{4}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

19. $\csc\theta = \frac{\sqrt{10}}{2}$, θ is Quadrant II

Answer. $\sin\theta = \frac{\sqrt{10}}{5}$, $\cos\theta = -\frac{\sqrt{6}}{\sqrt{10}}$, $\tan\theta = -\frac{\sqrt{6}}{3}$, $\sec\theta = -\frac{\sqrt{10}}{\sqrt{6}}$, $\cot\theta = -\frac{\sqrt{6}}{2}$

20. $\cos\theta = -\frac{5}{8}$, θ is Quadrant III

Answer. $\sin\theta = -\frac{\sqrt{39}}{8}$, $\tan\theta = \frac{\sqrt{39}}{5}$, $\csc\theta = -\frac{8}{\sqrt{39}}$, $\sec\theta = -\frac{8}{5}$, $\cot\theta = \frac{5}{\sqrt{39}}$

21. $\sec\theta = -2$, $\pi < \theta < \frac{3\pi}{2}$

Answer. $\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\csc\theta = -\frac{2}{\sqrt{3}}$, $\cot\theta = \frac{1}{\sqrt{3}}$

22. $\cot\theta = -\frac{5}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$

Answer. $\sin\theta = -\frac{3}{\sqrt{34}}$, $\cos\theta = \frac{5}{\sqrt{34}}$, $\tan\theta = -\frac{3}{5}$, $\csc\theta = -\frac{\sqrt{34}}{3}$, $\sec\theta = \frac{\sqrt{34}}{5}$

23. $\cos\theta = \frac{2}{3}$, $0 < \theta < \pi$

Answer. $\sin\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = \frac{\sqrt{5}}{2}$, $\csc\theta = \frac{3}{\sqrt{5}}$, $\sec\theta = \frac{3}{2}$, $\cot\theta = \frac{2}{\sqrt{5}}$

24. $\tan\theta = \frac{7}{4}$, $0 < \theta < \frac{\pi}{2}$

Answer. $\sin\theta = \frac{7}{\sqrt{65}}$, $\cos\theta = \frac{4}{\sqrt{65}}$, $\csc\theta = \frac{\sqrt{65}}{7}$, $\sec\theta = \frac{\sqrt{65}}{4}$, $\cot\theta = \frac{4}{7}$

25. $\csc\theta = \frac{3}{2}$, $\tan\theta < 0$

Answer. $\sin\theta = \frac{2}{3}$, $\cos\theta = -\frac{\sqrt{5}}{3}$, $\tan\theta = -\frac{2}{\sqrt{5}}$, $\sec\theta = -\frac{3}{\sqrt{5}}$, $\cot\theta = -\frac{\sqrt{5}}{2}$

26. $\sin\theta = \frac{5}{6}$, $\cot\theta > 0$

Answer. $\cos\theta = \frac{\sqrt{11}}{6}$, $\tan\theta = \frac{5}{\sqrt{11}}$, $\csc\theta = \frac{6}{5}$, $\sec\theta = \frac{6}{\sqrt{11}}$, $\cot\theta = \frac{\sqrt{11}}{5}$

27. $\sin\theta = -\frac{15}{17}$, $\cos\theta < 0$

Answer. $\cos\theta = -\frac{8}{17}$, $\tan\theta = \frac{15}{8}$, $\csc\theta = -\frac{17}{15}$, $\sec\theta = -\frac{17}{8}$, $\cot\theta = \frac{8}{15}$

28. $\cot\theta = -\frac{1}{3}$, $\sin\theta > 0$

Answer. $\sin\theta = \frac{3\sqrt{10}}{10}$, $\cos\theta = -\frac{\sqrt{10}}{10}$, $\tan\theta = -3$, $\csc\theta = \frac{\sqrt{10}}{3}$, $\sec\theta = -\sqrt{10}$

Exercise Group. Given a reference angle, t', calculate the corresponding angle, t, in standard position, along with the values of $\sin t$, $\cos t$, and $\tan t$ for:

(a) Quadrant II

(b) Quadrant III

(c) Quadrant IV

29. $t' = \frac{\pi}{4}$

Answer 1. $t = \frac{3\pi}{4}$, $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan\frac{3\pi}{4} = -1$

Answer 2. $t = \frac{5\pi}{4}$, $\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$, $\cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan\frac{5\pi}{4} = 1$

Answer 3. $t = \frac{7\pi}{4}$, $\sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$, $\cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}$, $\tan\frac{7\pi}{4} = -1$

30. $t' = \frac{\pi}{3}$

Answer 1. $t = \frac{2\pi}{3}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$

Answer 2. $t = \frac{4\pi}{3}$, $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} = -\frac{1}{2}$, $\tan\frac{4\pi}{3} = \sqrt{3}$

Answer 3. $t = \frac{5\pi}{3}$, $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{5\pi}{3} = \frac{1}{2}$, $\tan\frac{5\pi}{3} = -\sqrt{3}$

31. $t' = 30°$

Answer 1. $t = 150°$, $\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$, $\tan 150° = -\frac{1}{\sqrt{3}}$

Answer 2. $t = 210°$, $\sin 210° = -\frac{1}{2}$, $\cos 210° = -\frac{\sqrt{3}}{2}$, $\tan 210° = \frac{1}{\sqrt{3}}$

Answer 3. $t = 330°$, $\sin 330° = -\frac{1}{2}$, $\cos 330° = \frac{\sqrt{3}}{2}$, $\tan 330° = -\frac{1}{\sqrt{3}}$

32. $t' = 60°$

Answer 1. $t = 120°$, $\sin 120° = \frac{\sqrt{3}}{2}$, $\cos 120° = -\frac{1}{2}$, $\tan 120° = -\sqrt{3}$

Answer 2. $t = 240°$, $\sin 240° = -\frac{\sqrt{3}}{2}$, $\cos 240° = -\frac{1}{2}$, $\tan 240° = \sqrt{3}$

Answer 3. $t = 300°$, $\sin 300° = -\frac{\sqrt{3}}{2}$, $\cos 300° = \frac{1}{2}$, $\tan 300° = -\sqrt{3}$

Exercise Group. For each angle θ,

(a) determine the quadrant in which θ lies;

(b) calculate the reference angle θ';

(c) use the reference angle, θ', to evaluate the exact values of the six trigonometric functions for θ.

33. $\theta = -\frac{3\pi}{4}$

Answer 1. QIII

Answer 2. $\theta' = \frac{\pi}{4}$

Answer 3. $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\tan\theta = 1$, $\csc\theta = -\sqrt{2}$, $\sec\theta = -\sqrt{2}$, $\cot\theta = 1$

34. $\theta = \frac{4\pi}{3}$

Answer 1. QIII

Answer 2. $\theta' = \frac{\pi}{3}$

Answer 3. $\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\csc\theta = -\frac{2}{\sqrt{3}}$, $\sec\theta = -2$, $\cot\theta = \frac{1}{\sqrt{3}}$

35. $\theta = \frac{11\pi}{6}$

Answer 1. QIV

Answer 2. $\theta' = \frac{\pi}{6}$

Answer 3. $\sin\theta = -\frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\frac{1}{\sqrt{3}}$, $\csc\theta = -2$, $\sec\theta = \frac{2}{\sqrt{3}}$, $\cot\theta = -\sqrt{3}$

36. $\theta = \frac{7\pi}{3}$

Answer 1. QI

Answer 2. $\theta' = \frac{\pi}{3}$

Answer 3. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$, $\csc\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = 2$, $\cot\theta = \frac{1}{\sqrt{3}}$

37. $\theta = 120°$

Answer 1. QII

Answer 2. $\theta' = 60°$

Answer 3. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = -\sqrt{3}$, $\csc\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = -2$, $\cot\theta = -\frac{1}{\sqrt{3}}$

38. $\theta = 480°$

Answer 1. QII

Answer 2. $\theta' = 60°$

Answer 3. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = -\sqrt{3}$, $\csc\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = -2$, $\cot\theta = -\frac{1}{\sqrt{3}}$

Exercise Group. Use the fact that the trigonometric functions are periodic to find the exact value for each expression.

39. $\tan 420°$

Answer. $\sqrt{3}$

40. $\csc 540°$

Answer. Undefined

41. $\sin 765°$

Answer. $\frac{1}{\sqrt{2}}$

42. $\sec 1200°$

Answer. -2

43. $\cot \frac{8\pi}{3}$

Answer. $-\frac{1}{\sqrt{3}}$

44. $\cos \frac{21\pi}{4}$

Answer. $-\frac{1}{\sqrt{2}}$

45. $\tan \frac{35\pi}{6}$

Answer. $-\frac{1}{\sqrt{3}}$

46. $\sin \frac{39\pi}{4}$

Answer. $-\frac{1}{\sqrt{2}}$

47. Prove the second Pythagorean Identity (Definition 1.5.20): $1 + \tan^2\theta = \sec^2\theta$.

Hint. Begin with $\sin^2\theta + \cos^2\theta = 1$ and divide both sides of the equation by $\cos^2\theta$.

48. Prove the third Pythagorean Identity (Definition 1.5.20): $1 + \cot^2\theta = \csc^2\theta$.

Hint. Begin with $\sin^2\theta + \cos^2\theta = 1$ and divide both sides of the equation by $\sin^2\theta$.

Exercise Group. Use the Pythagorean Identity to find the exact value of the following:

49. $\sin^2 38° + \cos^2 38°$

Answer. 1

50. $\csc^2 13° - \cot^2 13°$

Answer. 1

51. $\cot^2 200° - \csc^2 200°$

Answer. -1

52. $\sec^2 \frac{6\pi}{5} - \tan^2 \frac{6\pi}{5}$

Answer. 1

53. $\tan^2 \frac{5\pi}{7} - \sec^2 \frac{5\pi}{7}$

Answer. -1

54. $\sin^2 \frac{14\pi}{13} + \cos^2 \frac{14\pi}{13}$

Answer. 1

Exercise Group. Use the Pythagorean Identities to express the first trigonometric function of θ in terms of the second function, given the quadrant.

55. $\sin\theta$, $\cos\theta$, Quadrant III

Answer. $\sin\theta = -\sqrt{1-\cos^2\theta}$

56. $\cos\theta$, $\sin\theta$, Quadrant II

Answer. $\cos\theta = -\sqrt{1-\sin^2\theta}$

57. $\tan\theta$, $\sec\theta$, Quadrant IV

Answer. $\tan\theta = -\sqrt{\sec^2\theta - 1}$

58. $\cot\theta$, $\csc\theta$, Quadrant III

Answer. $\cot\theta = \sqrt{\csc^2\theta - 1}$

59. $\tan\theta$, $\sin\theta$, Quadrant III

Answer. $\tan\theta = -\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

60. $\tan\theta$, $\cos\theta$, Quadrant II

Answer. $\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$

Exercise Group. Use the Pythagorean Identities to find the exact values of the remaining five trigonometric functions of θ from the given information.

61. $\tan\theta = -\frac{4}{3}$, θ is in Quadrant IV

Answer. $\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

62. $\cos\theta = -\frac{1}{4}$, θ is in Quadrant II

Answer. $\sin\theta = \frac{\sqrt{15}}{4}$, $\tan\theta = -\sqrt{15}$, $\csc\theta = \frac{4}{\sqrt{15}}$, $\sec\theta = -4$, $\cot\theta = -\frac{1}{\sqrt{15}}$

63. $\sin\theta = -\frac{2}{3}$, θ is in Quadrant III

Answer. $\cos\theta = -\frac{\sqrt{5}}{3}$, $\tan\theta = \frac{2}{\sqrt{5}}$, $\csc\theta = -\frac{3}{2}$, $\sec\theta = -\frac{3}{\sqrt{5}}$, $\cot\theta = \frac{\sqrt{5}}{2}$

64. $\cos\theta = \frac{3}{5}$, θ is in Quadrant IV

Answer. $\sin\theta = -\frac{4}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$

Exercise Group. Use the even and odd properties to evaluate the following:

65. $\cos(-60°)$

Answer. $\frac{1}{2}$

66. $\tan(-225°)$

Answer. -1

67. $\csc(-330°)$

Answer. 2

68. $\sin(-90°)$

Answer. -1

69. $\cot(-300°)$

Answer. $\frac{1}{\sqrt{3}}$

70. $\sec(-150°)$

Answer. $-\frac{2}{\sqrt{3}}$

71. $\sin\left(-\frac{11\pi}{6}\right)$

Answer. $\frac{1}{2}$

72. $\tan\left(-\frac{5\pi}{4}\right)$

Answer. -1

73. $\cos\left(-\frac{4\pi}{3}\right)$

Answer. $-\frac{1}{2}$

74. $\tan(-\pi)$

Answer. 0

75. $\sec\left(-\frac{\pi}{4}\right)$

Answer. $\sqrt{2}$

76. $\csc\left(-\frac{7\pi}{6}\right)$

Answer. 2

Exercise Group. In Remark 1.5.9, we learned that if the reference angle is given in radians and has the form $t' = \frac{a\pi}{b}$, then the corresponding angle in standard position, t, can be determined using an equation.

77. Prove that if t is in Quadrant II, then

$$t = \frac{(b-a)\pi}{b}.$$

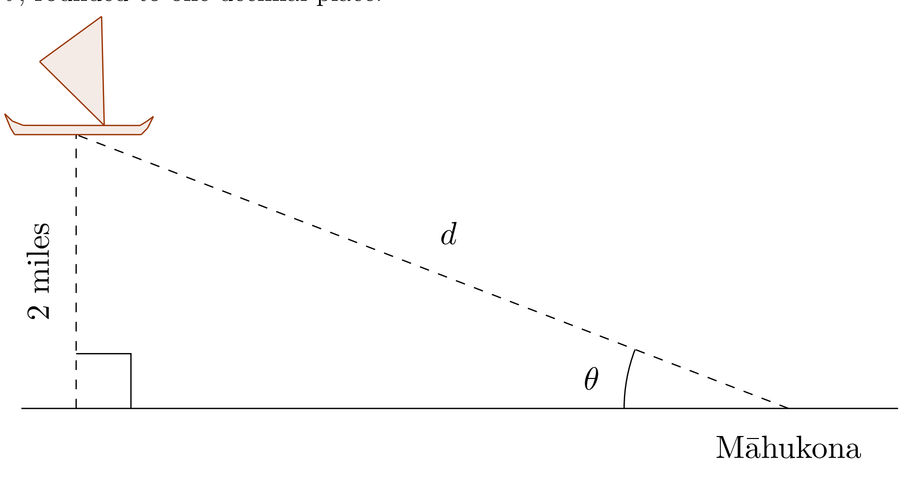
78. Prove that if t is in Quadrant III, then

$$t = \frac{(b+a)\pi}{b}.$$

79. Prove that if t is in Quadrant IV, then

$$t = \frac{2(b-a)\pi}{b}.$$

Exercise Group. The Makaliʻi is sailing along the Kohala Coast, maintaining a distance of two nautical miles from the shore. An observer at Māhukona is monitoring Makaliʻi's passage. Let d denote the length of the line connecting Makaliʻi to the Māhukona observer. Given θ as the angle formed between d and the shore, determine Makaliʻi's distance, d, from the observer for each value of θ, rounded to one decimal place.



80. $\theta = 30°$

Answer. 4 NM

81. $\theta = 45°$

Answer. 2.8 NM

82. $\theta = 60°$

Answer. 2.3 NM

83. $\theta = 90°$

Answer. 2 NM

84. $\theta = 120°$

Answer. 2.3 NM

85. $\theta = 135°$

Answer. 2.8 NM

86. $\theta = 150°$

Answer. 4 NM

Chapter 2

Graphs of the Trigonometric Functions

2.1 Graphs of the Sine and Cosine Functions

Sunrise and sunset are vital markers for navigational orientation and course corrections. During sunrise, we can determine the direction of wind and waves and observe their origin. As the sun rises higher, steering by the sun becomes impractical, and we must instead depend on swells to keep our course. At sunset, we can reassess your position and take note of any changes in wind and swell patterns. Although the general pattern is for the sun to rise in the east and set in the west, its exact position on the horizon, known as **solar declination**, varies throughout the year.

To understand why the position changes, we must first learn about Earth's **axial tilt**, which represents the angle between the Earth's rotational axis and its orbital plane around the Sun. To conceptualize this tilt, envision a pole passing through the Earth's center, extending from the North to the South Pole, with the Earth revolving around this axis. Each complete rotation of the Earth on this axis corresponds to one day. As the Earth travels along its orbit around the Sun with a constant tilt of the axis, the orientation of the Earth's axial tilt causes either the North Pole or the South Pole to tilt toward the Sun. This tilt varies depending on the Earth's location in its orbit relative to the Sun.

During the equinoxes, which mark the transition from winter to spring and from summer to fall, the Earth's axis is not tilted towards or away from the Sun. Consequently, on these days, the Sun rises due east and sets due west, and the duration of day and night is approximately equal.

Following the **fall equinox** in the Southern Hemisphere, typically occurring around March 20th, the Earth proceeds along its orbit around the Sun, leading the Southern Hemisphere to tilt away from the Sun. As a result, the Sun's rising position progressively moves northward each day. By the time of the **winter solstice**, around June 20th, the Sun rises from its northernmost position, resulting in the shortest day in the Southern Hemisphere and the longest day of the year in the Northern Hemisphere.

After the winter solstice, the Earth's axial tilt remains the same, but the Southern Hemisphere starts tilting toward the Sun. This causes the Sun's rising position to gradually shift southward each day. When the **spring equinox** arrives, approximately on September 22nd, the Sun rises due east once again, and day and night are once more of equal duration.

As the Southern Hemisphere continues to tilt towards the Sun, the Sun's rising position moves even further south each day. By the time of the **summer solstice**, approximately on December 21st, the Sun rises from its southernmost position, resulting in the longest day of the year in the Southern Hemisphere and the shortest day in the Northern Hemisphere.

Following the summer solstice, the Southern Hemisphere begins tilting away from the Sun, causing the Sun's rising position to gradually shift northward each day. This movement continues until the next fall equinox, completing the annual cycle.

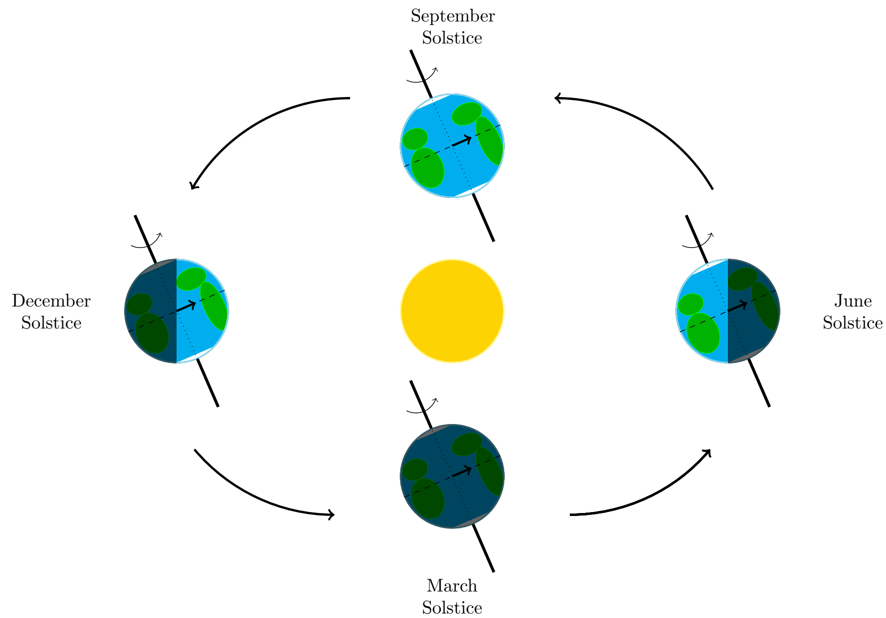


Figure 2.1.1 The illustration shows the relative positions and timing of solstice, equinox and seasons in relation to the Earth's orbit around the Sun and the axial tilt. During the June solstice, the northern hemisphere is tilted towards the Sun, while the southern hemisphere is tilted away. Conversely, during the December solstice, the southern hemisphere is tilted towards the Sun, and the northern hemisphere is tilted away. During the March and September equinoxes, the Earth's axis is perpendicular to the line connecting the Earth to the Sun, causing the Sun to appear directly above the equator. As a result, day and night are approximately equal in length all over the world.

The position where the Sun rises throughout the year exhibits a repeating pattern, characterized as a **periodic function** with a cycle of about one year. This function can be mathematically represented as either a sine or cosine function. Graphically, it illustrates the sunrise position on the horizon relative to the east (Hikina) as time progresses. By examining this function, we can observe the gradual northward and southward movement of the sunrise position throughout the year. Key dates such as the equinoxes and solstices mark significant points in this pattern, providing insights into the changing seasons and variations in daylight hours.

In this section, we will explore the periodic nature of sine and cosine functions and study their transformations. Studying the graphs of sine and cosine functions provides valuable insights into the world around us. The graphs of the remaining trigonometric functions will be covered in Section 2.2.

2.1.1 Domain and Range of Sine and Cosine

The domains for $\sin(\theta)$ and $\cos(\theta)$ consist of the set of the inputs for the functions. Since any angle θ can be input into sine and cosine and still have these functions defined, the domain for both sine and cosine is all real numbers. Recall that in Section 1.3, if $P(x, y)$ is any point on the unit circle that corresponds to the angle θ, we defined $\sin\theta = y$ and $\cos\theta = x$. Given the constraints on the unit circle, $-1 \le x \le 1$ and $-1 \le y \le 1$, and thus

$$-1 \le \sin\theta \le 1, \quad \text{and} \quad -1 \le \cos\theta \le 1.$$

Since the range of a function consists of all its outputs, we conclude that the range of both the sine and cosine functions spans all real numbers between -1 and 1.

Remark 2.1.2 Domain and Range for the Sine and Cosine Functions. Table 2.1.3 summarizes the domains and ranges for the sine and cosine functions.

Table 2.1.3 Domains and Ranges of Sine and Cosine

Function	Domain	Range
$\sin\theta$	All real numbers, $(-\infty, \infty)$	All real numbers from -1 to 1, $[-1, 1]$
$\cos\theta$	All real numbers, $(-\infty, \infty)$	All real numbers from -1 to 1, $[-1, 1]$

2.1.2 The Sine Function

Convention 2.1.4 In Chapter 1, trigonometric functions typically use θ or t as the variable in the domain, such as $y = \cos\theta$ and $y = \sin t$. However, when graphing functions on the Cartesian plane (xy-coordinate system), x is conventionally used as the variable in the domain. Therefore, when graphing trigonometric functions, we will use x as the variable, for example $y = \cos x$ and $y = \sin x$.

The sine function, as discussed in Subsection 1.5.4, is a periodic functions with period 2π. To graph $y = \sin x$, we can focus on the interval $[0, 2\pi]$. By plotting the points on the graph over this interval, we can then repeat the values over the entire domain to complete the graph.

Recall from Definition 1.3.2 that on the unit circle, $\sin\theta$ is defined to be the y-value of the terminal point $P(x, y)$ on the unit circle associated with the angle θ. As the angle increases from 0 to $\frac{\pi}{2}$, the y-value also increases from 0 to 1. When the angle continues from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the y-value decreases from 1 to -1. "Finally, as the angle increases from $\frac{3\pi}{2}$ to 2π, the y-value increases from -1 to 0. This behavior is shown in Figure 2.1.5.

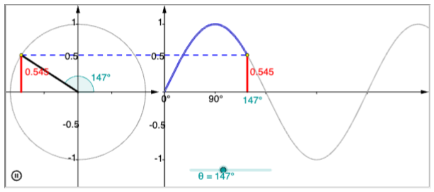


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Figure 2.1.5 As θ moves from $0°$ to $360°$, this figure plots the values of $y = \sin\theta$. Move the slider for θ to see how changing the angle affects $\sin\theta$. Note that while we will generally be using radians when graphing trigonometric functions, this figure uses degrees to help visualize the angle. For example, we may not be aware that 2 radians ($\approx 114.6°$) lies in Quadrant II. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

Next we recall known values for the sine function listed in Table 2.1.6.

Table 2.1.6 Values for $y = \sin x$.

x	$\sin x$	(x, y)	x	$\sin x$	(x, y)
0	0	$(0, 0)$	π	0	$(\pi, 0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6}, \frac{1}{2}\right)$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\left(\frac{4\pi}{3}, -\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$	$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$
			2π	0	$(2\pi, 0)$

Now that we have a visual understanding of the graph for $y = \sin x$, we can utilize the data from Table 2.1.6 to map out the points. This process enables us to construct the graph illustrated in Figure 2.1.7, representing one complete period of the sine function.

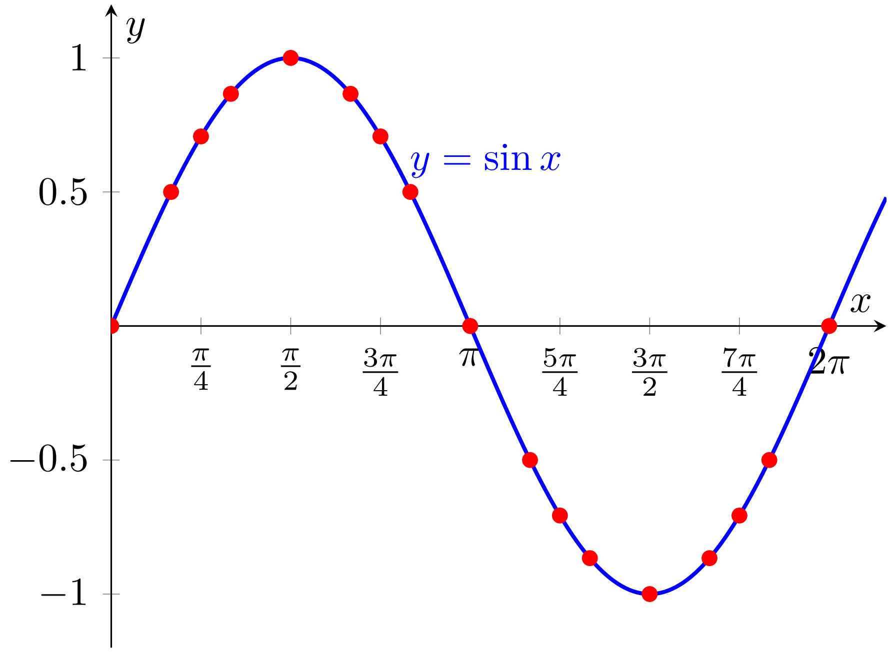

Figure 2.1.7 The values in Table 2.1.6 plotted with a smooth curve connecting the points to make the graph for $y = \sin(x)$.

Since the graph in Figure 2.1.7 represents one period, we can now complete the graph of $y = \sin x$ by extending the pattern in both directions to obtain Figure 2.1.8.

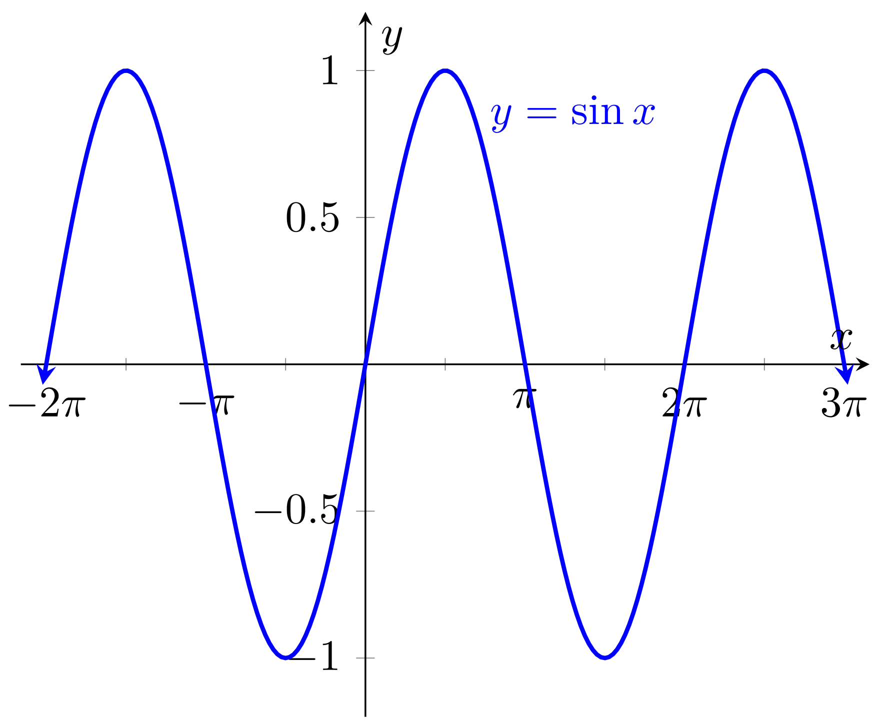

Figure 2.1.8 The graph for $y = \sin(x)$.

Notice the sine function's symmetry with respect to the origin, a characteristic supported in Section 1.5.7, where we learned that sine is an odd function.

2.1.3 The Cosine Function

Similarly, we can construct a plot for the cosine function, shown in Figure 2.1.9.

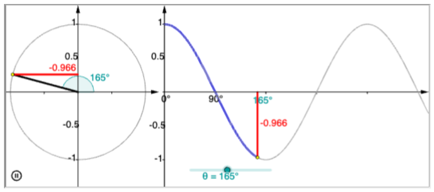

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Figure 2.1.9 As θ moves from $0°$ to $360°$, this figure plots the values of $y = \cos\theta$. Move the slider for θ to see how changing the angle affects $\cos\theta$. Note that while we will generally be using radians when graphing trigonometric functions, this figure uses degrees to help visualize the angle. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

By plotting points for $y = \cos x$ and using the fact that the cosine function is periodic, we obtain the graph for cosine over the entire domain. This is shown in Figure 2.1.10.

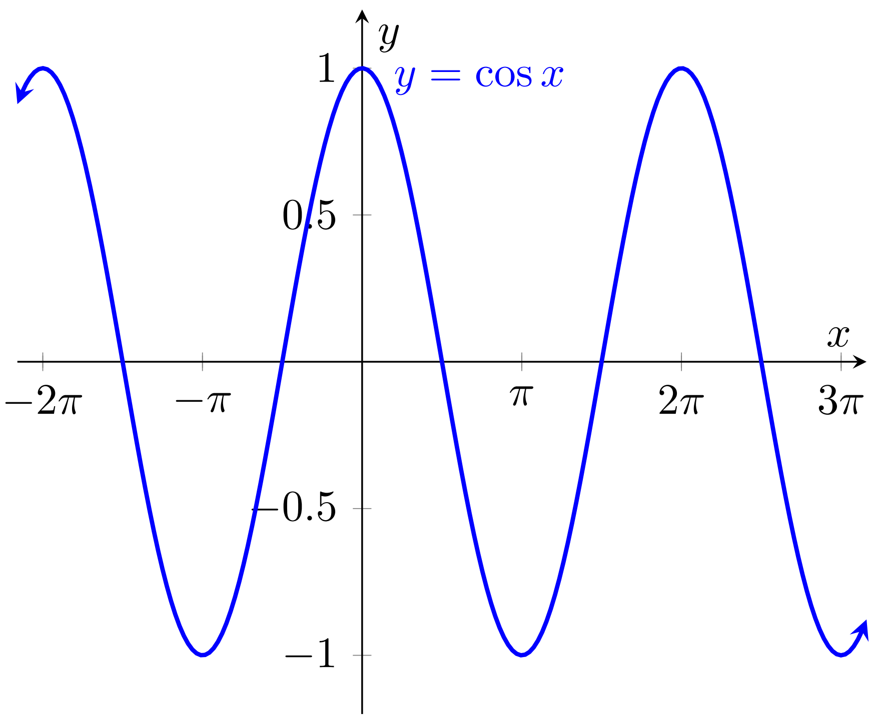


Figure 2.1.10 The graph for $y = \cos(x)$.

In alignment with Section 1.5.7, observe that the graph of cosine is symmetric about the y-axis, confirming that it is an even function.

Definition 2.1.11 The graphs for the sine and cosine functions are commonly referred to as **sinusoidal graphs** or **sinus curves**. $\lozenge$

2.1.4 Graphing Transformations of Sine and Cosine

Now that we've become familiar with the graphs of the sine and cosine functions, let's apply algebraic graphing techniques to these functions. Recall that when

$D > 0$, the graph of $y = f(x) + D$ shifts the graph of $y = f(x)$ upward by D units, and the graph of $y = f(x) - D$ shifts the graph of $y = f(x)$ downward by D units.

Definition 2.1.12 Vertical Shift. The graphs of the functions

$$y = \sin x + D \quad \text{and} \quad y = \cos x + D$$

represent an **upward vertical shift** of the graphs $y = \sin(x)$ and $y = \cos(x)$ by D units, respectively.

Similarly, the functions

$$y = \sin x - D \quad \text{and} \quad y = \cos x - D$$

depict the graphs of $y = \sin(x)$ and $y = \cos(x)$ with a **downward vertical shift** by D units, respectively. ◊

Example 2.1.13 Vertical Shifts. Graph the functions

(a) $y = \sin(x) + 2$

Solution.

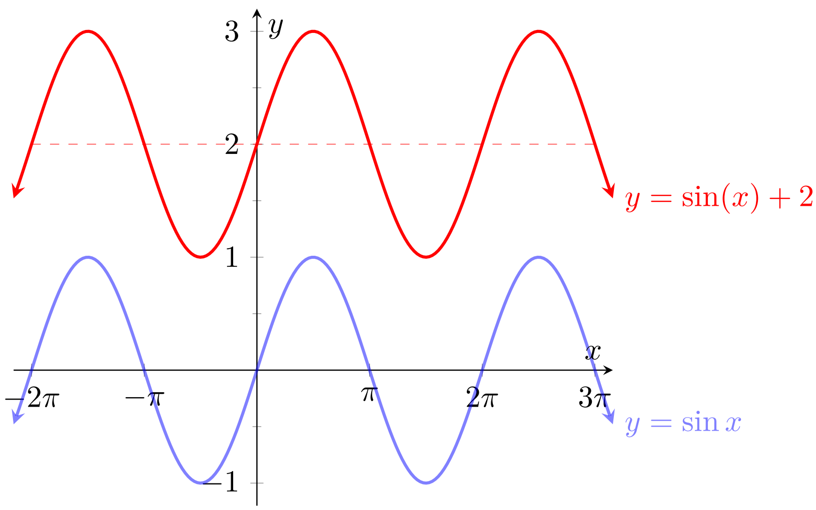


Figure 2.1.14 The graph of $y = \sin(x) + 2$ is the same as the graph of $y = \sin(x)$ but shifted up by 2 units.

(b) $y = \cos(x) - 1$

□

Additionally, remember that the graph $y = -f(x)$ reflects the graph of $y = f(x)$ about the x-axis.

Definition 2.1.15 Reflection about the x-axis. The functions

$$y = -\sin x \quad \text{and} \quad y = -\cos x$$

represent the graphs of $y = \sin(x)$ and $y = \cos(x)$ with a **reflection about the x-axis**, respectively. ◊

Example 2.1.16 Reflections about the x-axis. Graph $y = -\cos(x)$

Solution.

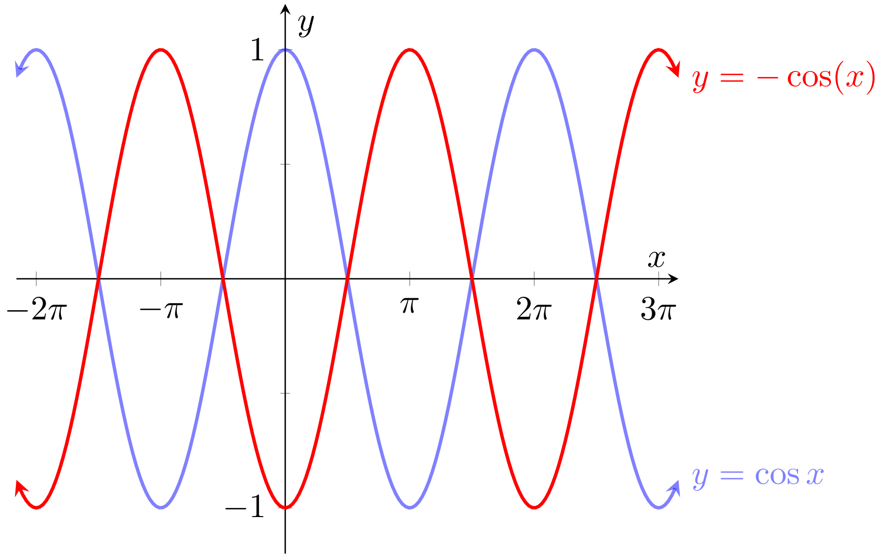


Figure 2.1.17 The graph $y = -\cos(x)$ is obtained by multiplying every y-value of the $y = \cos(x)$ graph by -1. This transformation reflects all points across the x-axis, turning positive values negative and negative values positive.

□

Similarly, recall that the graph of $y = f(-x)$ reflects the graph of $y = f(x)$ about the y-axis.

Definition 2.1.18 Reflection about the y-axis. The functions

$$y = \sin(-x) \quad \text{and} \quad y = \cos(-x)$$

represent the graphs of $y = \sin(x)$ and $y = \cos(x)$ with a **reflection about the y-axis**, respectively. ◊

Example 2.1.19 Reflections about the y-axis. Graph $y = \sin(-x) + 1$.

Solution.

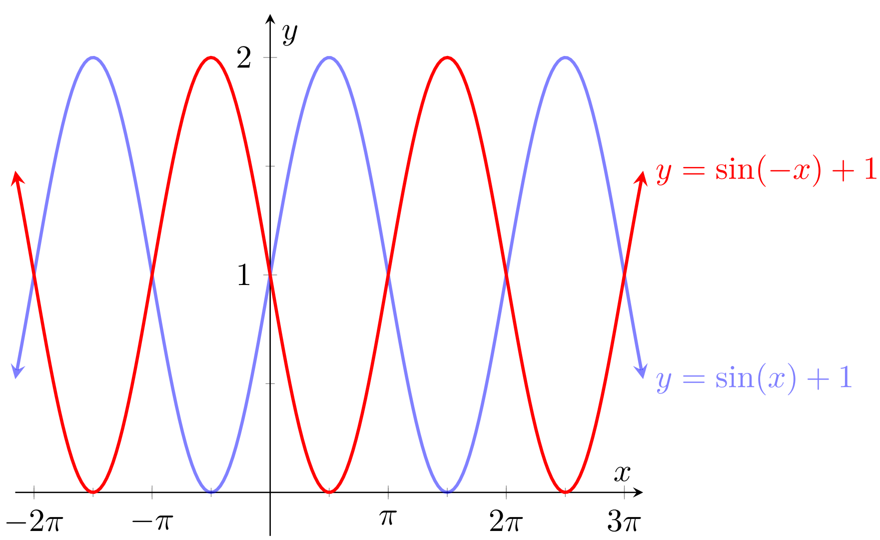


Figure 2.1.20 The graph $y = \sin(-x) + 1$ is obtained by reflecting the graph of $y = \sin(x)$ about the y-axis and then vertically shifting it upward by 1. This transformation turns positive values of x negative and negative values of x positive, and increases every y-value by 1.

□

Example 2.1.21 Vertical Stretches and Compressions. Graph each function:

1. $y = 2\cos(x)$
2. $y = \dfrac{1}{2}\cos(x)$

Solution.

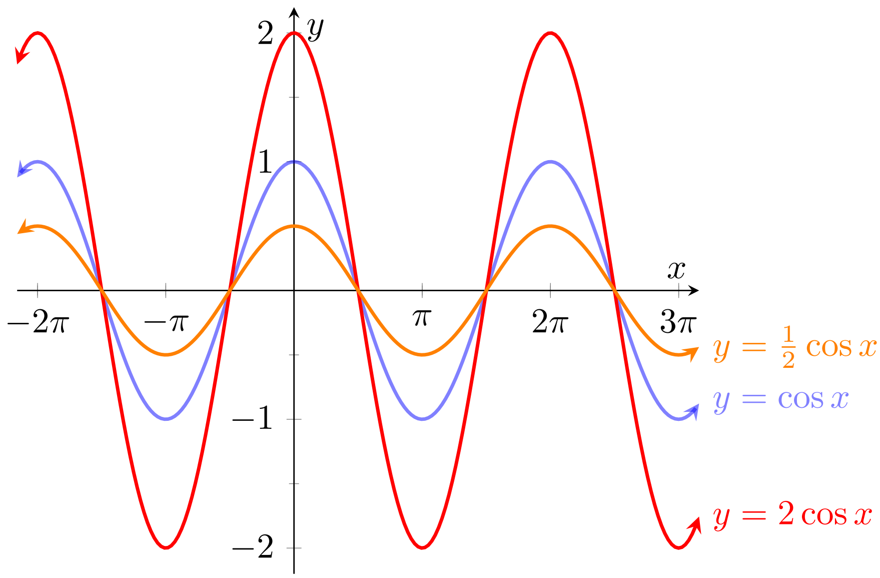


Figure 2.1.22 The graph of $y = 2\cos(x)$ is achieved by vertically stretching the y-values of $y = \cos(x)$ by a factor of 2. Similarly, the graph of $y = \frac{1}{2}\cos(x)$ is obtained by vertically compressing the y-values of $y = \cos(x)$ by a factor of $\frac{1}{2}$.

□

The factor multiplied at the front of the cosine function plays a crucial role in stretching and compressing the graph. This factor is known as the **amplitude**, measuring the maximum vertical distance from the midline to the peak or trough of a sinusoidal wave. In Example 2.1.21, the amplitude of $y = 2\cos(x)$ is 2, indicating a **vertical stretch** by a factor of 2 compared to the standard cosine function. Conversely, for $y = \frac{1}{2}\cos(x)$, the amplitude is $\frac{1}{2}$, representing a **vertical compression** by a factor of $\frac{1}{2}$.

Definition 2.1.23 Amplitude. For a sinusoidal function, the **amplitude**, denoted as $|A|$, is the height of the function, representing half the distance between its maximum and minimum values:

$$|A| = \text{amplitude} = \frac{\text{maximum} - \text{minimum}}{2}.$$

In other words, the amplitude is the vertical distance from the midline to the maximum or minimum value of the function. The **midline** is a horizontal line representing the average value of the function. It can be calculated by:

$$y = \frac{\text{maximum} + \text{minimum}}{2}.$$

For a graph oscillating symmetrically about the x-axis, the amplitude is simply the maximum value of the function.

In general, for

$$y = A \cdot \sin(x) \quad \text{or} \quad y = A \cdot \cos(x),$$

the amplitude is given by $|A|$. This absolute value ensures that amplitude is always a positive value, representing the magnitude of the vertical stretching or compression. ◊

Definition 2.1.24 Vertical Stretch/Compression. The functions

$$y = A \sin x \quad \text{and} \quad y = A \cos x$$

represents a sine and cosine function, respectively, with an amplitude of $|A|$. The amplitude determines the vertical stretch or compression of the graph.

If $|A| > 1$, the graph undergoes a vertical stretch, increasing the distance between the peaks and troughs.

If $0 < |A| < 1$, the graph undergoes a vertical compression, reducing the distance between the peaks and troughs. ◊

Example 2.1.25 Graph $y = -4 \sin x$ and identify the amplitude.

Solution. The amplitude is $|-4| = 4$.

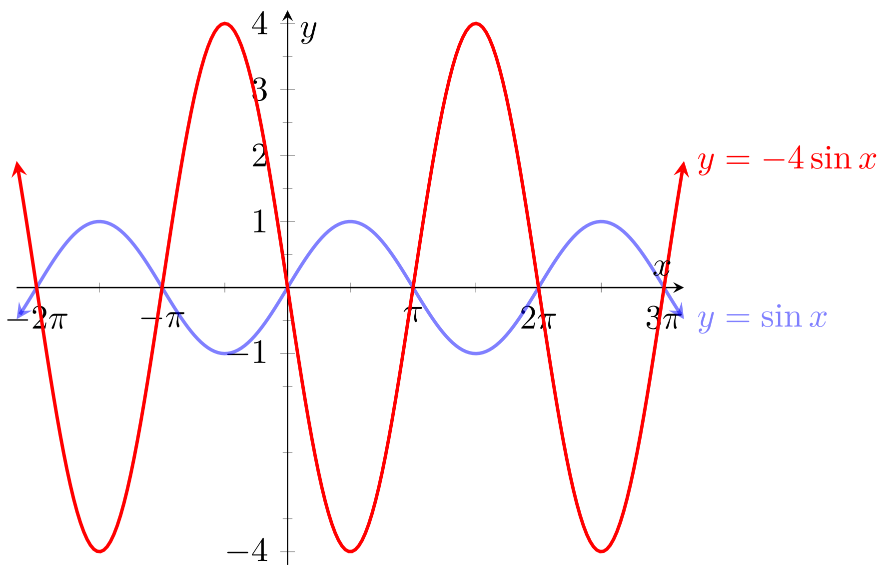


Figure 2.1.26 Since the amplitude of $y = -4\sin(x)$ is 4, the graph is stretched by a factor of 4 and will oscillate between -4 and 4. Additionally, the negative sign indicates that the graph is reflected about the x-axis.

□

Next we will look at functions of the form

$$y = \sin Bx \quad \text{and} \quad y = \cos Bx.$$

Recall from algebra that for functions of the form $y = f(Bx)$, a key factor emerges: when $|B| > 1$, the graph undergoes horizontal compression by a factor of $\frac{1}{|B|}$; conversely, when $0 < |B| < 1$, the graph is horizontally stretched by a factor of $\frac{1}{|B|}$. Given that the period of sine and cosine is 2π, the horizontal stretching or compressing of a period will be by a factor of $\frac{1}{|B|}$.

Definition 2.1.27 Period. For sine and cosine functions of the form

$$y = \sin Bx \quad \text{and} \quad y = \cos Bx$$

the **period** is defined as

$$\text{period} = \frac{2\pi}{|B|}.$$

Thus, if $|B| > 1$, the period is compressed; if $0 < |B| < 1$, the period is stretched. ♢

Definition 2.1.28 Horizontal Stretch/Compression. The graphs functions

$$y = \sin Bx \quad \text{and} \quad y = \cos Bx,$$

undergo a horizontal stretch or compression by a value of $\frac{1}{|B|}$.

If $|B| > 1$, the graph undergoes a horizontal compression, making the period shorter.

If $0 < |B| < 1$, the graph undergoes a horizontal stretch, making the period longer. ♢

Example 2.1.29 Horizontal Stretches and Compressions. Identify the period and graph one period for each of the following functions:

1. $y = \sin(2x)$
2. $y = \sin\left(\dfrac{1}{2}x\right)$
3. $y = \sin\left(\dfrac{1}{3}x\right)$

Solution.

1. The period for $y = \sin(2x)$ is $\frac{2\pi}{2} = \pi$.
2. The period for $y = \sin\left(\frac{1}{2}x\right)$ is $\frac{2\pi}{\frac{1}{2}} = 2\pi \cdot \frac{2}{1} = 4\pi$.
3. The period for $y = \sin\left(\frac{1}{3}x\right)$ is $\frac{2\pi}{\frac{1}{3}} = 2\pi \cdot \frac{3}{1} = 6\pi$.

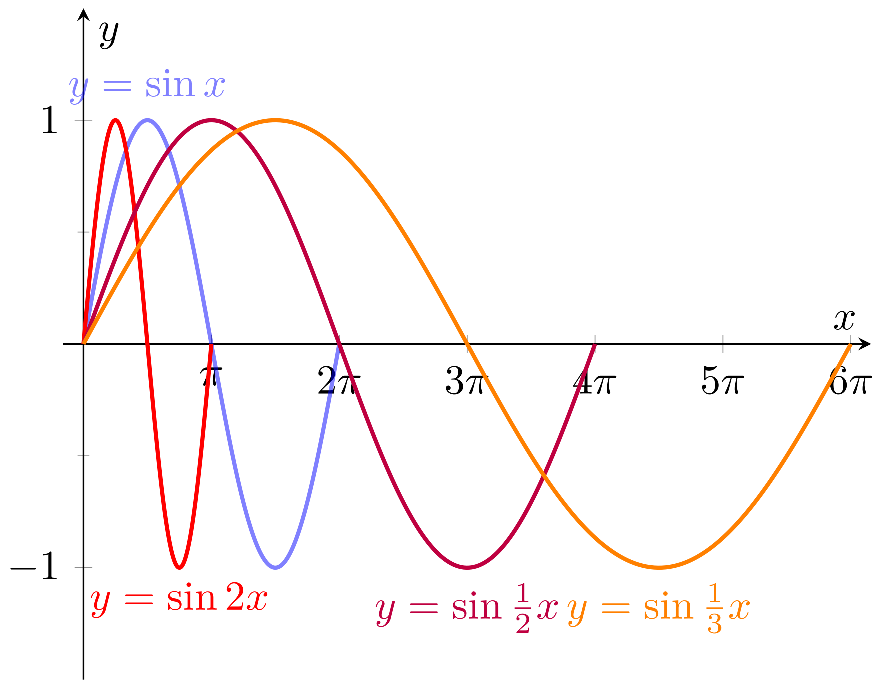


Figure 2.1.30 One period of each of $y = \sin(2x)$, $y = \sin\left(\frac{1}{2}x\right)$, and $y = \sin\left(\frac{1}{3}x\right)$ compared to one period of the graph of $y = \sin(x)$. Observe the distinct effects of horizontal compression (when $B = 2$, reducing the period to π) and stretching (when $B = \frac{1}{2}$ and $B = \frac{1}{3}$, increasing the period to 4π and 6π, respectively).

□

Our final transformation involves functions of the form $y = f(x - c)$. When $c > 0$, the graph of $y = f(x)$ is shifted by c units to the right; when $c < 0$, it is shifted $|c|$ units to the left.

Definition 2.1.31 Phase Shift. The functions

$$y = \sin(B(x - C)) \quad \text{and} \quad y = \cos(B(x - C)) \tag{2.1.1}$$

undergo a **horizontal shift**, known as **phase shift**, of C units. If $C > 0$, the phase shift is to the right; if $C < 0$, it is to the left. ◊

Remark 2.1.32 Functions of the form $y = \sin(Bx - E)$ and $y = \cos(Bx - E)$. Note that you may see functions written in the form

$$y = \sin(Bx - E) \quad \text{and} \quad y = \cos(Bx - E). \tag{2.1.2}$$

There is a subtle yet important difference between (2.1.1) and (2.1.2). In (2.1.1), the term B, affecting the period, is multiplied by both x and C, the phase shift. In (2.1.2), B is only multiplied by x. We can rewrite (2.1.2) by factoring out B as

$$y = \sin\left(B\left(x - \frac{E}{B}\right)\right) \quad \text{and} \quad y = \cos\left(B\left(x - \frac{E}{B}\right)\right).$$

This form aligns with (2.1.1). Therefore, for equations of the form

$$y = \sin(Bx - E) \quad \text{and} \quad y = \cos(Bx - E)$$

the phase shift is $\frac{E}{B}$ units.

Example 2.1.33 Phase Shift. Identify the period and phase shift of each function, and graph the function.

(a) $y = \cos(x - \pi)$

Solution. Since this equation is of the form $y = \cos(Bx - E)$, we have $B = 1$ and $E = \pi$. Therefore,

$$\text{period} = \frac{2\pi}{|B|} = \frac{2\pi}{1} = 2\pi$$

and

$$\text{phase shift} = \frac{E}{B} = \frac{\pi}{1} = \pi.$$

A positive value for the phase shift indicates a shift to the right. It's important to note that, given $B = 1$, the phase shift is simply $E = \pi$.

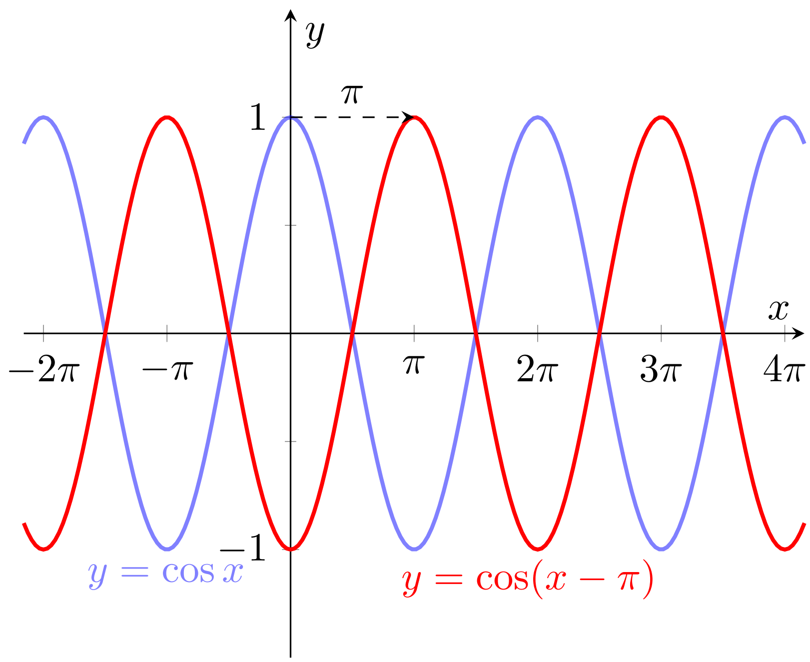


Figure 2.1.34 The graph of $y = \cos(x - \pi)$ is the graph of $y = \cos x$ with a phase shift of π units to the right.

(b) $y = \sin\left(\frac{\pi}{6}(x + 2)\right)$

Solution. Since the given equation is in the form $y = \sin\left(\frac{\pi}{6}(x + 2)\right)$, we can identify $B = \frac{\pi}{6}$ and $C = -2$. Consequently,

$$\text{period} = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \frac{6}{\pi} = 12$$

and

$$\text{phase shift} = C = -2.$$

The negative sign indicates a phase shift to the left by 2 units.

To graph $y = \sin\left(\frac{\pi}{6}(x+2)\right)$, begin by graphing the sine function $y = \sin\left(\frac{\pi}{6}x\right)$ with a period of 12. Then, apply a phase shift of 2 units to the left on the resulting graph.

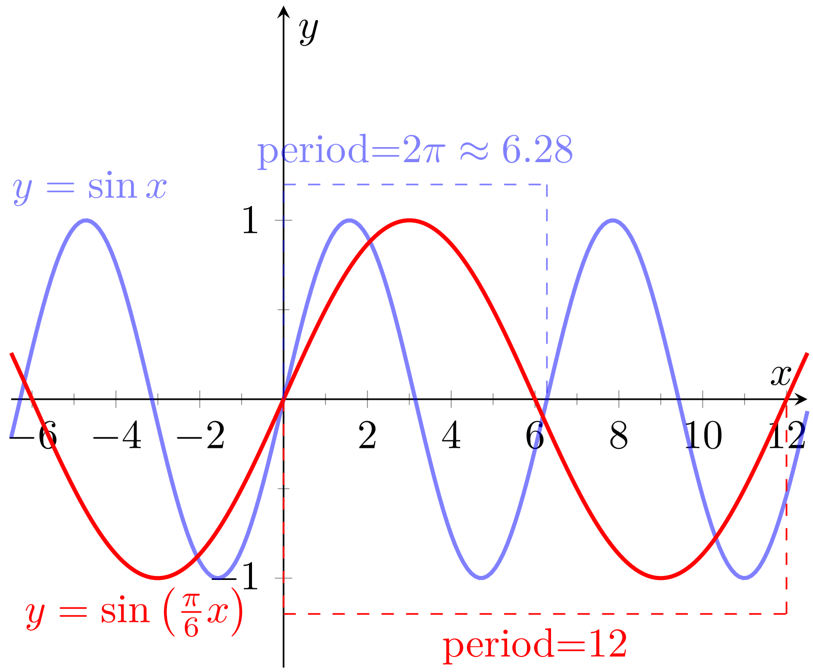


Figure 2.1.35 The graph of $y = \sin\left(\frac{\pi}{6}x\right)$ represents the sine function $y = \sin x$ with a horizontal stretch, resulting in a period of 12.

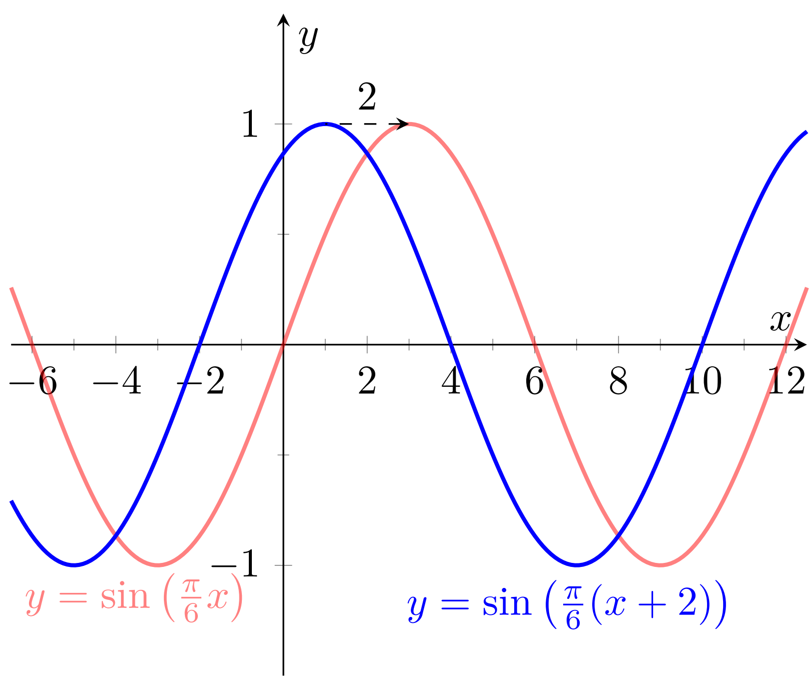


Figure 2.1.36 Shifting the graph of $y = \sin\left(\frac{\pi}{6}x\right)$ by two units to the left results in the graph of $y = \sin\left(\frac{\pi}{6}(x+2)\right)$.

□

We will now summarize the transformations by consolidating them into a single equation.

Remark 2.1.37 Transformations of Sine and Cosine. When dealing with functions in the form

$$y = A\sin(B(x-C)) + D \quad \text{and} \quad y = A\cos(B(x-C)) + D$$

we can express the transformations as follows:

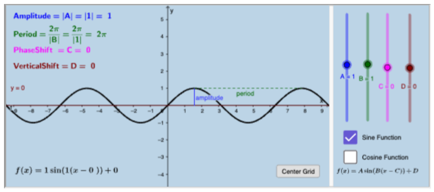
- Amplitude and Vertical Stretch/Compression: $|A|$
  - $|A|$ is the value of the amplitude.
  - If $|A| > 1$, there is vertical stretching.
  - If $0 < |A| < 1$, there is vertical compression.
- Period and Horizontal Stretch/Compression: $|B|$
  - The period is $\frac{2\pi}{|B|}$.
  - If $|B| > 1$, there is horizontal compression and the period is shortened.
  - If $0 < |B| < 1$, there is horizontal stretching and the period is lengthened.
- Phase Shift: C
  - If C is positive, there is a shift to the right.
  - If C is negative, there is a shift to the left.
- Vertical Shift: D
  - If D is positive, there is a shift upward.
  - If D is negative, there is a shift downward.
- Reflection about the x-axis:
  - If A is negative ($A < 0$), there is a reflection about the x-axis.
- Reflection about the y-axis:
  - If B is negative ($B < 0$), there is a reflection about the y-axis.

Remark 2.1.38 Transformations of the form $y = A \cdot \sin(Bx - E) + D$ and $y = A \cdot \cos(Bx - E) + D$. For functions of the form

$$y = A\sin(Bx - E) + D \quad \text{and} \quad y = A\cos(Bx - E) + D$$

the transformations are the same as above, except for the phase shift where we replace C with $\frac{E}{B}$.

Explore the effects of various transformations using the interactive features in Figure 2.1.39.



Standalone

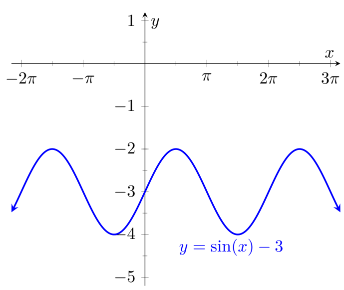
Figure 2.1.39 Manipulate the graphs of sine and cosine by adjusting the sliders for A, B, C, and D. Observe the effects on amplitude, period, phase and vertical shifts, as well as reflections about the x- and y-axes. Additionally, we can toggle between the sine and cosine graphs by selecting the corresponding function. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the “Standalone” link to explore the interactive version online.

2.1.5 Exercises

Exercise Group. Graph the function.

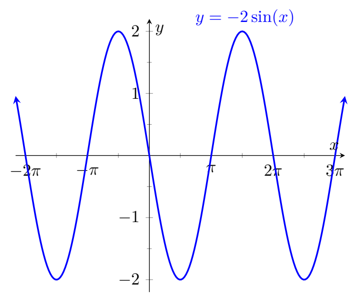
1. $y = \sin(x) - 3$

Answer.



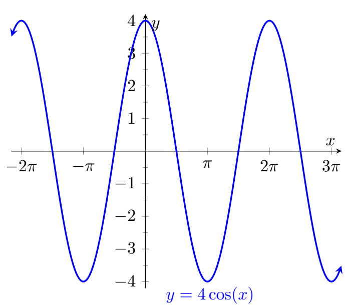
2. $y = -2\sin(x)$

Answer.



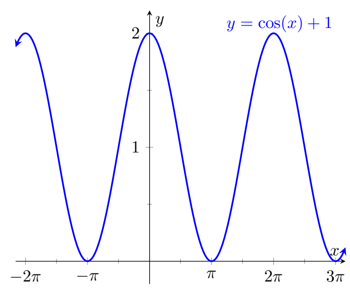
3. $y = 4\cos(x)$

Answer.



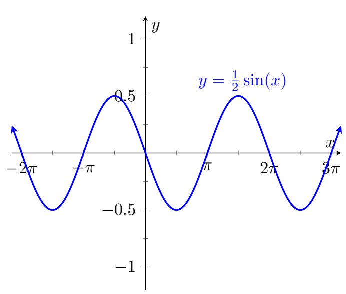
4. $y = \cos(x) + 1$

Answer.



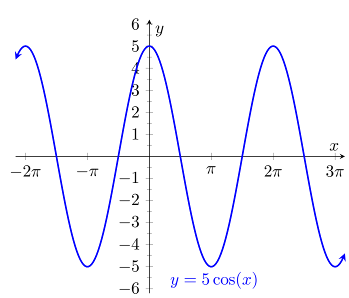
5. $y = \frac{1}{2}\sin(x)$

Answer.



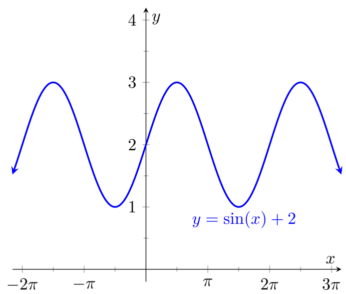
6. $y = 5\cos(x)$

Answer.



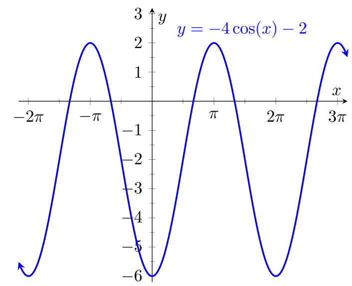
7. $y = \sin(x) + 2$

Answer.



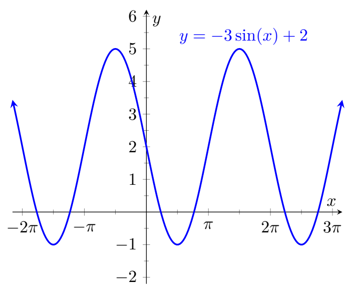
8. $y = -4\cos(x) - 2$

Answer.



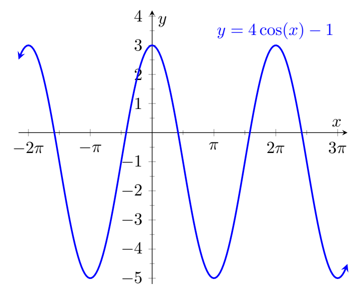
9. $y = -3\sin(x) + 2$

Answer.



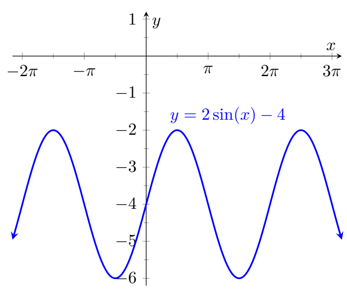
10. $y = 4\cos(x) - 1$

Answer.



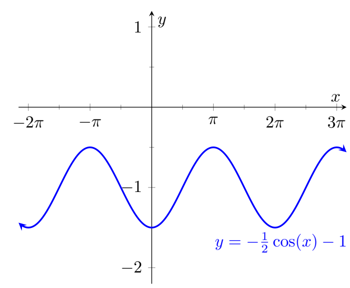
11. $y = 2\sin(x) - 4$

Answer.



12. $y = -\frac{1}{2}\cos(x) - 1$

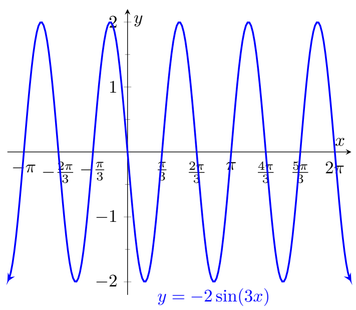
Answer.



Exercise Group. Determine the amplitude and period for each function, and sketch the graph.

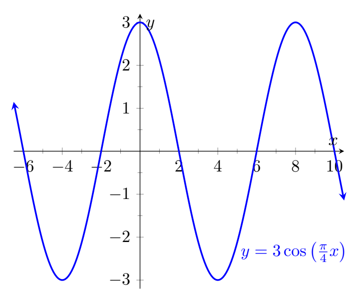
13. $y = -2\sin(3x)$

Answer. Amplitude: 2; Period: $\frac{2\pi}{3}$;



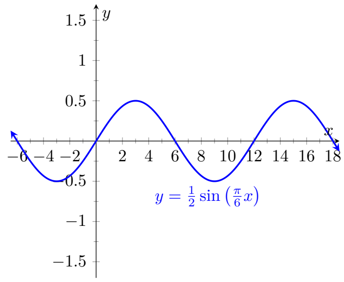
14. $y = 3\cos\left(\frac{\pi}{4}x\right)$

Answer. Amplitude: 3; Period: 8;



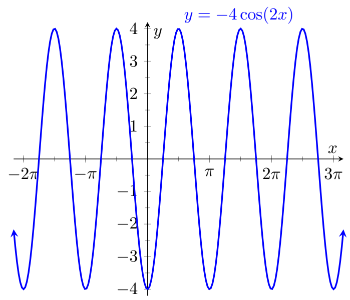
15. $y = \frac{1}{2}\sin\left(\frac{\pi}{6}x\right)$

Answer. Amplitude: $\frac{1}{2}$; Period: 12;



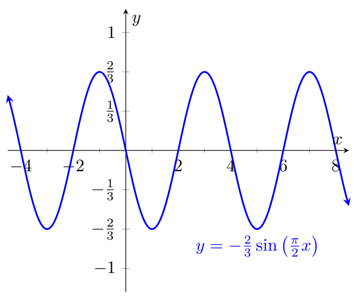
16. $y = -4\cos(2x)$

Answer. Amplitude: 4; Period: π;



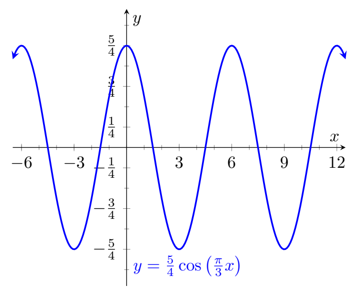
17. $y = -\frac{2}{3}\sin\left(\frac{\pi}{2}x\right)$

Answer. Amplitude: $\frac{2}{3}$; Period: 4;



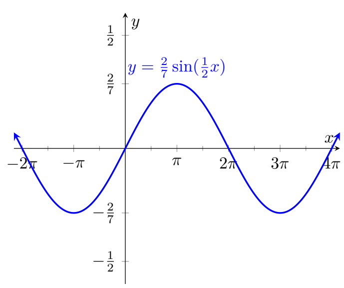
18. $y = \frac{5}{4}\cos\left(\frac{\pi}{3}x\right)$

Answer. Amplitude: $\frac{5}{4}$; Period: 6;



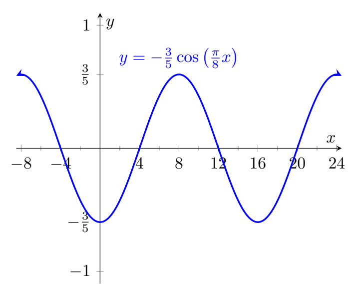
19. $y = \frac{2}{7}\sin(\frac{1}{2}x)$

Answer. Amplitude: $\frac{2}{7}$; Period: 4π;

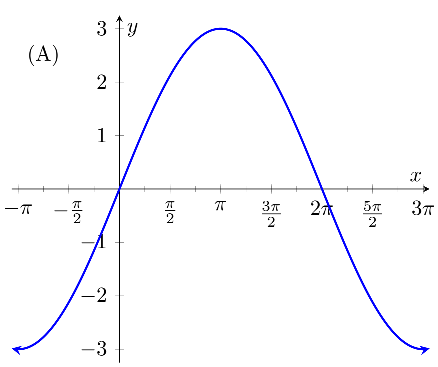


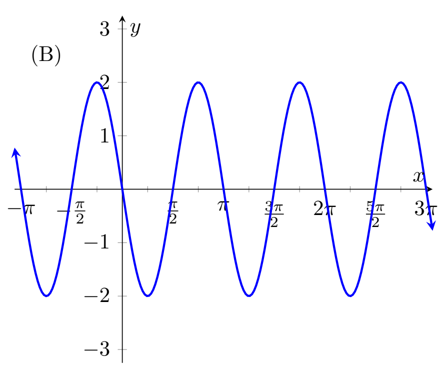
20. $y = -\frac{3}{5}\cos\left(\frac{\pi}{8}x\right)$

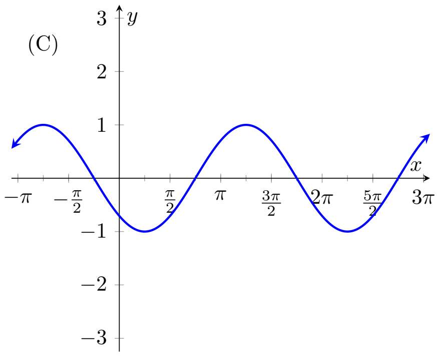
Answer. Amplitude: $\frac{3}{5}$; Period: 16;

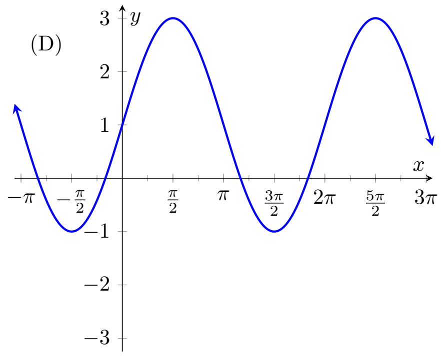


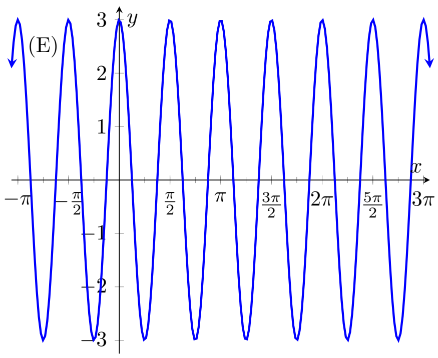
Exercise Group. Match the given function to one of the graphs below.

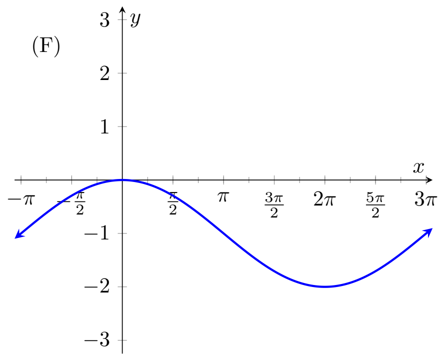












21. $y = 2\sin(x) + 1$

Answer. D

22. $y = 3\cos(4x)$

Answer. E

23. $y = -2\sin(2x)$

Answer. B

24. $y = 3\sin\left(\frac{1}{2}x\right)$

Answer. A

25. $y = -\cos\left(x - \frac{\pi}{4}\right)$

Answer. C

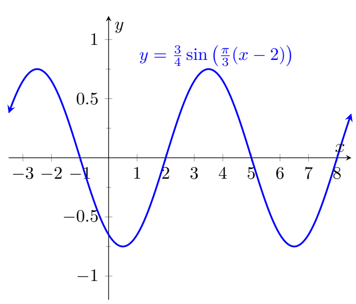
26. $y = \cos\left(\frac{1}{2}x\right) - 1$

Answer. F

Exercise Group. Find the amplitude, period, phase shift, and vertical shift of each function and sketch the graph.

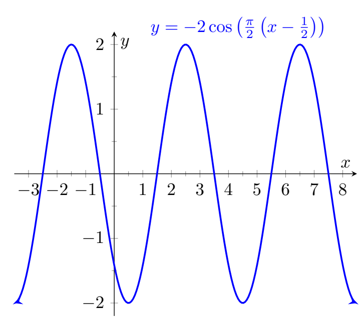
27. $y = \frac{3}{4}\sin\left(\frac{\pi}{3}(x-2)\right)$

Answer. Amplitude: $\frac{3}{4}$; Period: 6; Phase Shift: 2; Vertical Shift: 0



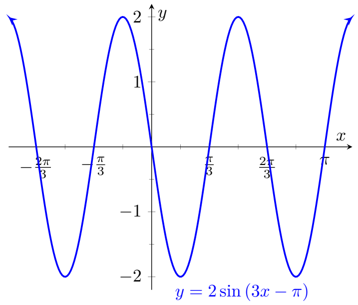
28. $y = -2\cos\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)$

Answer. Amplitude: 2; Period: 4; Phase Shift: 0.5; Vertical Shift: 0



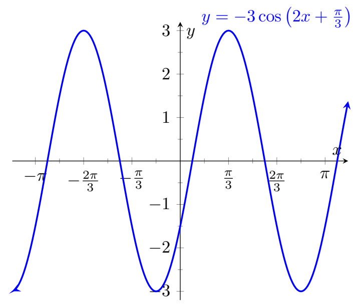
29. $y = 2\sin(3x - \pi)$

Answer. Amplitude: 2; Period: $\frac{2\pi}{3}$; Phase Shift: $\frac{\pi}{3}$; Vertical Shift: 0



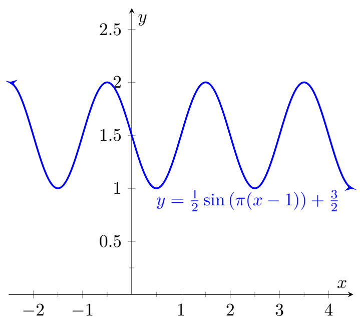
30. $y = -3\cos\left(2x + \frac{\pi}{3}\right)$

Answer. Amplitude: 3; Period: π; Phase Shift: $-\frac{\pi}{6}$; Vertical Shift: 0



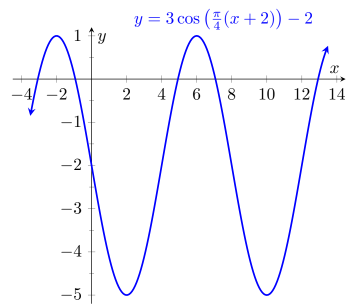
31. $y = \frac{1}{2}\sin(\pi(x - 1)) + \frac{3}{2}$

Answer. Amplitude: $\frac{1}{2}$; Period: 2; Phase Shift: 1; Vertical Shift: $\frac{3}{2}$



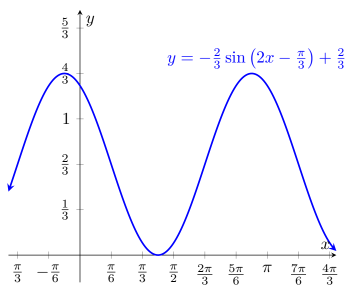
32. $y = 3\cos\left(\frac{\pi}{4}(x + 2)\right) - 2$

Answer. Amplitude: 3; Period: 8; Phase Shift: -2; Vertical Shift: -2



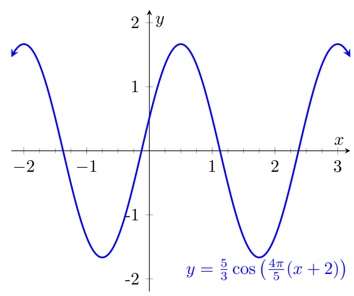
33. $y = -\frac{2}{3}\sin\left(2x - \frac{\pi}{3}\right) + \frac{2}{3}$

Answer. Amplitude: $\frac{2}{3}$; Period: 2; Phase Shift: $\frac{\pi}{6}$; Vertical Shift: $\frac{2}{3}$



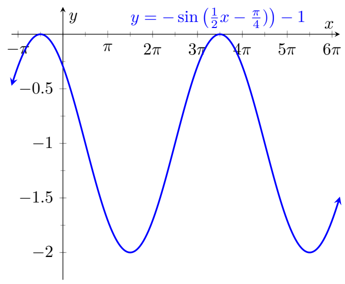
34. $y = \frac{5}{3}\cos\left(\frac{4\pi}{5}(x + 2)\right)$

Answer. Amplitude: $\frac{5}{3}$; Period: 2.5; Phase Shift: -2; Vertical Shift: 0



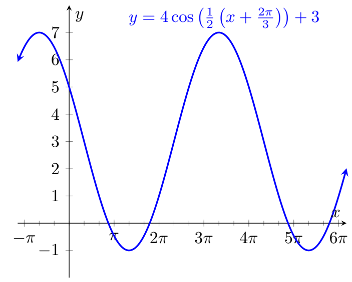
35. $y = -\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right) - 1$

Answer. Amplitude: 1; Period: 4π; Phase Shift: $\frac{\pi}{2}$; Vertical Shift: -1



36. $y = 4\cos\left(\frac{1}{2}\left(x + \frac{2\pi}{3}\right)\right) + 3$

Answer. Amplitude: 4; Period: 4π; Phase Shift: $-\frac{2\pi}{3}$; Vertical Shift: 3



Solar Declination. At the start of this section, we explored the effects of axial tilt on Earth's seasons, considering the sun's declination—the angle between the equator and a line drawn from the center of the Earth to the center of the Sun. When observing the sunrise, the sun's declination is the angle from the sunrise to due east.

During the June Solstice, the declination is $\delta = 23.45°$, causing the sun to rise 23.45° to the north of east. On the December Solstice, the declination is $\delta = -23.45°$, resulting in the sun rising 23.45° to the south of east.

Conversely, during the spring and fall equinox when the declination is $\delta = 0°$, the sun rises precisely at due east. This period holds significant historical importance, as observers, both in the past and present, have utilized this time to precisely determine the eastward direction from their positions. This practice is widespread across various cultures, with individuals using the equinox to establish directional markers. Homes, ceremonial sites, and other notable locations are often intentionally oriented based on the equinox.

In terms of navigation, knowing the solar declination angle for a specific time of year allows observers to measure the same angle down or up from where the sun rose during sunrise, thereby determining the direction of east.

To approximate the solar declination angle δ in degrees, we can use the following equation derived in [2.1.6.1]

$$\delta = -23.45° \cdot \cos\left(\frac{360}{365} \cdot (N + 10)\right)$$

where N represents the day of the year, with January 1 denoted as $N = 1$, and December 31 as $N = 365$.

For each of the following problems, calculate the solar declination for the given day, assuming a 365-day year. Round your answer to two decimals.

37. March 22nd (81st day of the year).

Answer. $\delta = -0.10°$

38. June 21st (172nd day of the year).

Answer. $\delta = 23.45°$

39. September 21st (264th day of the year).

Answer. $\delta = -0.10°$

40. December 21st (355th day of the year).

Answer. $\delta = -23.45°$

41. April 1st (91st day of the year).

Answer. $\delta = 3.92°$

42. September 3rd (246th day of the year).

Answer. $\delta = 7.05°$

43. May 28th (148th day of the year).

Answer. $\delta = 21.40°$

44. November 23rd (327th day of the year).

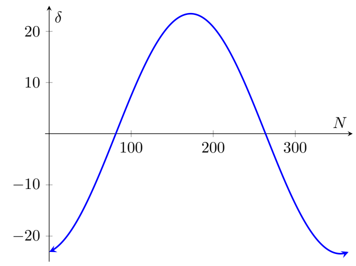
Answer. $\delta = -20.78°$

45. What is significant about March 22nd, June 21st, September 21st, and December 21st?

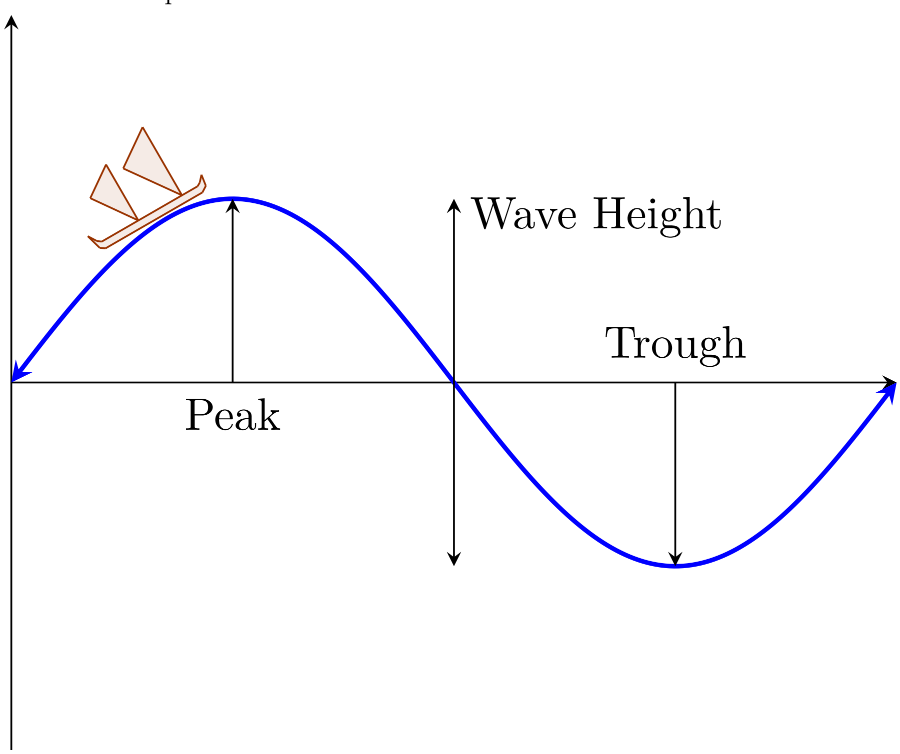
Answer. They are the equinoxes and soltices.

46. Graph the solar declination angle over time. Use the horizontal axis for N, the day of the year, and the vertical axis for δ, representing the solar declination angle in degrees.

Answer.

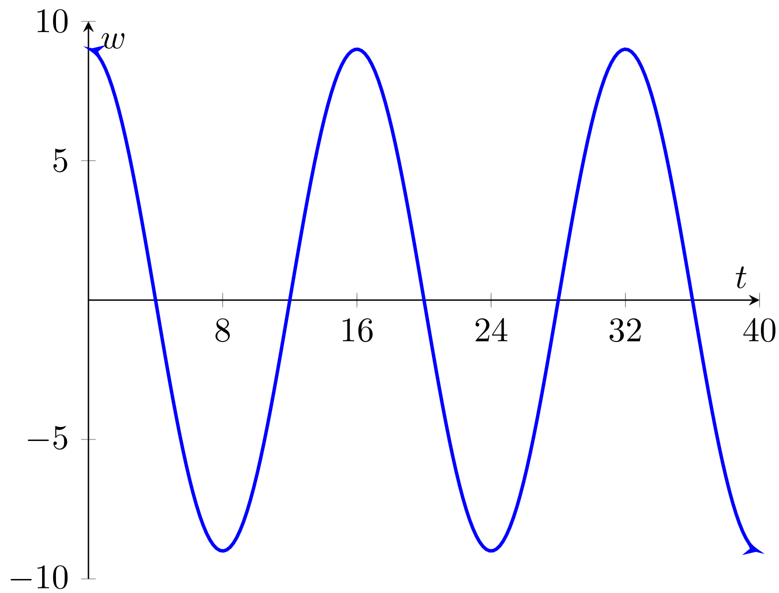


Rough Seas. Wave heights, defined as the vertical distance between the crest and the trough of a wave, can vary in the open ocean. However, the height of the wave alone does not necessarily indicate calm or choppy sailing conditions. Another important factor is the **wave period**, representing the time between waves, which affects the smoothness of sailing. For each given equation, where $w(t)$ is the number of feet the wave is above the mean sea level at t seconds, calculate: a) the wave height; b) the wave period; and plot the wave for two periods.



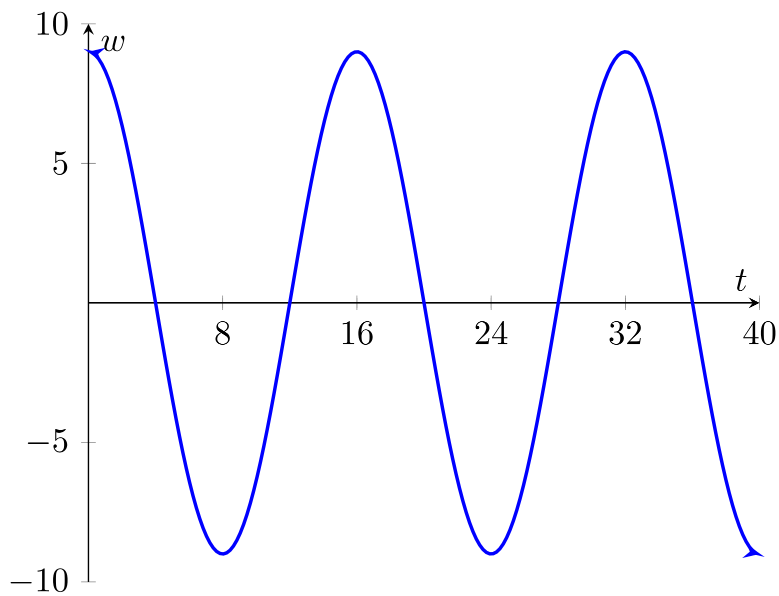
47. $w(t) = 4\cos\left(\frac{\pi}{8}t\right)$

Answer. a) 8 feet; b) 16 seconds



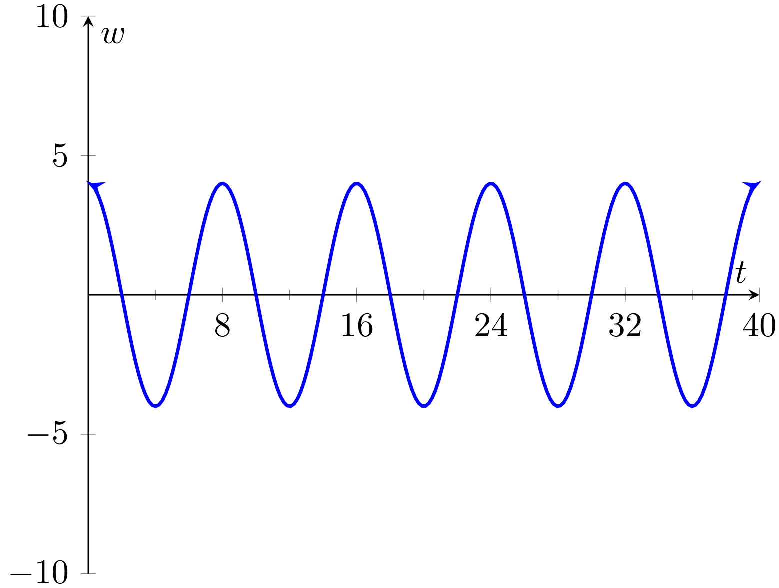
48. $w(t) = 9\cos\left(\frac{\pi}{8}t\right)$

Answer. a) 18 feet; b) 16 seconds



49. $w(t) = 4\cos\left(\frac{\pi}{4}t\right)$

Answer. a) 8 feet; b) 8 seconds



50. Which of the three waves above would give the smoothest sailing?

Answer. $w(t) = 4\cos\left(\frac{\pi}{8}t\right)$

2.1.6 References

[1] A. E. Dixon and J. D. Leslie, *Solar Energy Conversion*, Pergamon; (1979)

2.2 Graphs of Other Trigonometric Functions

This section explores the graphs of tangent, cotangent, cosecant, and secant, including their periodic behaviors and transformations.

2.2.1 Domains of the Tangent and Cotangent Functions

Recall the Quotient Identity for tangent (Definition 1.4.3):

$$\tan x = \frac{\sin x}{\cos x}.$$

$\tan x$ is undefined when the denominator is zero, that is, when $\cos x = 0$. This leads to undefined points at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$. In general, any angle of the form $n\frac{\pi}{2}$, where n is an odd integer, should be excluded from the domain since tangent is undefined at these values.

Similarly, we defined the cotangent function as

$$\cot x = \frac{\cos x}{\sin x}.$$

The denominator becomes zero when $\sin x = 0$, corresponding to $x = \ldots, -\pi, 0, \pi, 2\pi, \ldots$. In general, $\cot x$ is undefined for angles of the form $n\pi$, where n is an integer. These angles should be excluded from the domain of cotangent.

Remark 2.2.1 Domains for Tangent and Cotangent. The domains for tangent and cotangent functions are given in Table 2.2.2.

Table 2.2.2 Domains of the tangent and cotangent functions.

Function	Domain
$\tan\theta$	All real numbers except odd integer multiples of $\frac{\pi}{2}$ $(90°)$
$\cot\theta$	All real numbers except integer multiples of π $(180°)$

2.2.2 Ranges of the Tangent and Cotangent Functions

To determine the range of the tangent function, consider the point $P(x, y)$ on the unit circle corresponding to the angle θ, and let a be a real number such that $a = \tan\theta = \frac{y}{x}$.

Multiplying both sides by x, we obtain:

$$y = ax.$$

Squaring both sides yields:

$$y^2 = a^2x^2.$$

Substituting into the Pythagorean Identity (Definition 1.5.20), we have:

$$1 = x^2 + y^2 = x^2 + a^2x^2 = x^2(1 + a^2).$$

Dividing both sides by $1 + a^2$ and taking the square root gives:

$$x = \pm\frac{1}{\sqrt{1+a^2}}.$$

Similarly, we obtain:

$$y = \pm\frac{a}{\sqrt{1+a^2}}.$$

Thus, we conclude:

$$\tan\theta = \frac{y}{x} = \frac{\frac{a}{\sqrt{1+a^2}}}{\frac{1}{\sqrt{1+a^2}}} = a.$$

In other words, since a can be any real number and $\tan\theta = a$, the range of the tangent function consists of all real numbers. A similar method can be used to show that the range of the cotangent function is also the set of all real numbers.

Remark 2.2.3 Ranges for Tangent and Cotangent. The ranges for tangent and cotangent functions are given in Table 2.2.4.

Table 2.2.4 Ranges of the tangent and cotangent functions.

Function	Range
$\tan x$	All real numbers
$\cot x$	All real numbers

2.2.3 Domains of the Cosecant and Secant Functions

Consider the Reciprocal Identity defining the cosecant function (Definition 1.4.2):

$$\csc x = \frac{1}{\sin x}.$$

When $\sin x = 0$, corresponding to $x = \ldots, -\pi, 0, \pi, 2\pi, \ldots$, the denominator becomes zero. Therefore, $\csc x$ is undefined for angles of the form $n\pi$, where n is an integer, and these values should be excluded from the domain.

Similarly, since the secant function is defined as

$$\sec x = \frac{1}{\cos x}$$

we see that $\sec x$ is undefined when $\cos x = 0$. This occurs at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, and thus any angle of the form $n\frac{\pi}{2}$, where n is an odd integer, should be excluded from the domain of $\sec x$.

Remark 2.2.5 Domains for Cosecant and Secant. The domains for cosecant and secant functions are given in Table 2.2.6.

Table 2.2.6 Domains of the trigonometric functions.

Function	Domain
$\csc x$	All real numbers except integer multiples of π (180°)
$\sec x$	All real numbers except odd integer multiples of $\frac{\pi}{2}$ (90°)

2.2.4 Ranges of the Cosecant and Secant Functions

If the angle is not an integer multiple of π, i.e. $x \neq n\pi$, where n is an integer, then cosecant is defined by the Reciprocal Identity as

$$\csc x = \frac{1}{\sin x}.$$

In Subsection 2.1.1, we learned that the function $y = \sin x$ has a range of

$$-1 \leq \sin x \leq 1.$$

Therefore, taking the reciprocal of the range of sine, we get

$$\csc x \leq -1 \text{ or } \csc x \geq 1.$$

In other words, the range of the cosecant function is all real numbers less than or equal to -1 or greater than or equal to 1.

Similarly, since $-1 \leq \cos x \leq 1$, we can get the range for the secant function as

$$\sec x \leq -1 \text{ or } \sec x \geq 1$$

or all real numbers less than or equal to -1 or greater than or equal to 1.

Remark 2.2.7 Ranges for Cosecant and Secant. The ranges for cosecant and secant functions are given in Table 2.2.8.

Table 2.2.8 Ranges of the cosecant and secant functions.

Function	Range
$\csc x$	All real numbers less than or equal to -1 or greater than or equal to 1
$\sec x$	All real numbers less than or equal to -1 or greater than or equal to 1

2.2.5 The Tangent Function

In Subsection 1.5.4, we learned that the tangent function is periodic with a period of π. To graph $y = \tan x$, we focus on plotting the graph for one period and then repeating those values to complete the graph.

We also know that the domain of tangent includes all real numbers except angles of the form $n\frac{\pi}{2}$, where n is an odd integer. These values are excluded since tangent is undefined for these angles. In fact, any line of the form $x = n\frac{\pi}{2}$, where n is an odd integer (e.g., $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$), is a **vertical asymptote**.

Knowing the location of the vertical asymptotes, we choose the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to plot our points for tangent. This interval has a length of π (one period), allowing us to repeat the values to complete the graph of tangent over its entire domain.

Recall from Definition 1.3.2 that on the unit circle, the expression $\tan\theta = \frac{y}{x}$ denotes the ratio of the y-coordinate to the x-coordinate of a point $P(x, y)$ associated with an angle θ. This ratio changes as θ varies from zero to $\frac{\pi}{2}$.

As θ approaches zero, the y-value tends to zero, and x approaches 1, resulting in $\tan\theta$ being a small fraction. Conversely, as θ approaches $\frac{\pi}{2}$, the y-value approaches 1, while x becomes extremely small and approaches zero. This causes $\tan\theta$ to be evaluated as a fraction with a very small number in the denominator, producing a large number. A similar effect occurs when θ is between $-\frac{\pi}{2}$ and zero, except $\tan\theta$ is negative in this range. This behavior is shown in Figure 2.2.9.

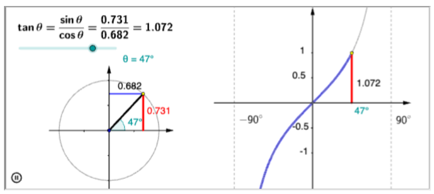


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Figure 2.2.9 As θ moves from $-90°$ to $90°$, this figure plots the values of $y = \tan\theta$. Move the slider for θ to see how changing the angle affects $\tan\theta$. Note that while we will generally be using radians when graphing trigonometric functions, this figure uses degrees to help visualize the angle. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

Next we recall values of tangent for known angles, which are listed in Table 2.2.10 and plotted in Figure 2.2.11.

Table 2.2.10 Values for $y = \tan x$.

x	$\tan x$	(x, y)
$-\frac{\pi}{3}$	$-\sqrt{3}$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{4}$	-1	$\left(-\frac{\pi}{4}, -1\right)$
$-\frac{\pi}{6}$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{\pi}{6}, -\frac{1}{\sqrt{3}}\right)$
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$	$\left(\frac{\pi}{6}, \frac{1}{\sqrt{3}}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\sqrt{3}$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$

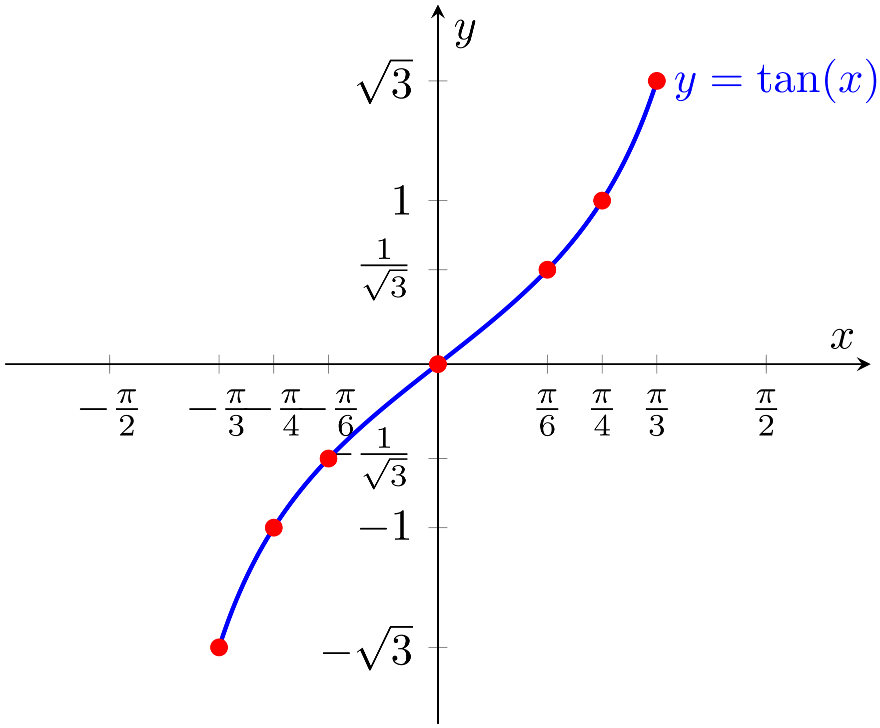

Figure 2.2.11 The values in Table 2.2.10 plotted with a smooth curve connecting the points to make the graph for $y = \tan(x)$.

Notice the symmetrical nature of the tangent function's graph with respect to the origin, a feature explained in Section 1.5.7, where we learned that tangent is an odd function.

Since the graph in Figure 2.2.11 represents one period, we can complete the graph of $y = \tan x$ by extending the pattern in both directions to obtain Figure 2.2.12.

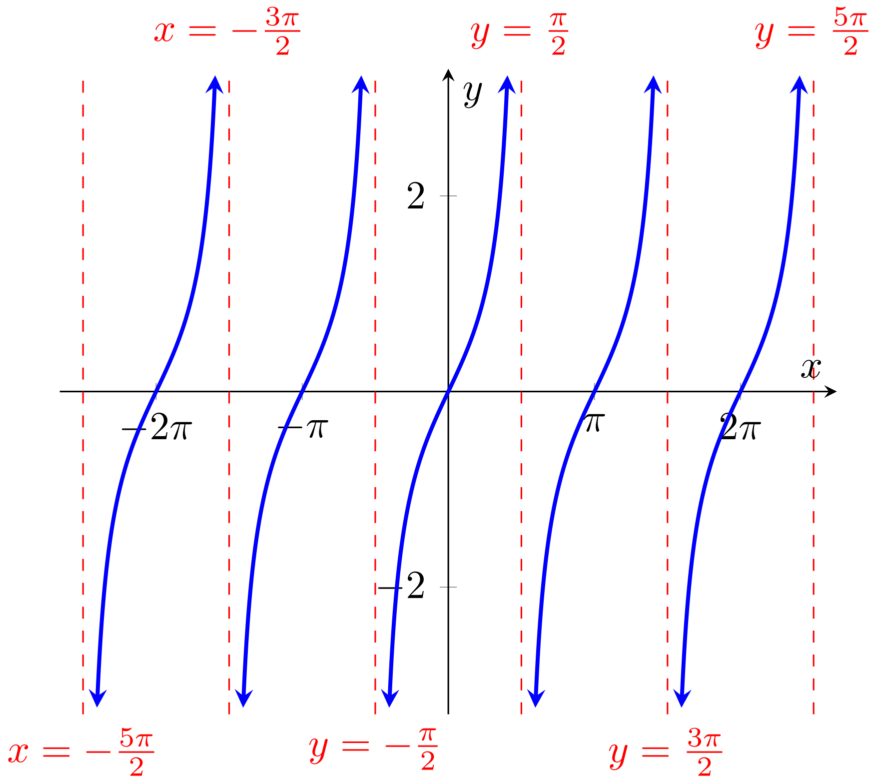

Figure 2.2.12 The graph for $y = \tan(x)$.

2.2.6 The Cotangent Function

The domain of the cotangent function contains all angles except those of the form $n\pi$, where n is an integer. These excluded values correspond to vertical asymptotes. In fact, any line of the form $x = n\pi$, where n is an integer, serves as a vertical asymptote. Additionally, as we learned in Subsection 1.5.4, the cotangent function has a period of π. We can construct a plot for it in a manner similar to how we constructed the tangent function, as illustrated in Figure 2.2.13.

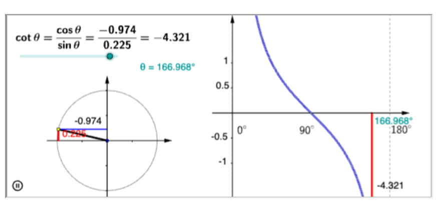

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Figure 2.2.13 As θ moves from $0°$ to $180°$, this figure plots the values of $y = \cot\theta$. Move the slider for θ to see how changing the angle affects $\cot\theta$. Note that while we will generally be using radians when graphing trigonometric functions, this figure uses degrees to help visualize the angle. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

Plotting points for $y = \cot x$ and using the fact that the cotangent function is periodic, we obtain the graph for cotangent in Figure 2.2.14.

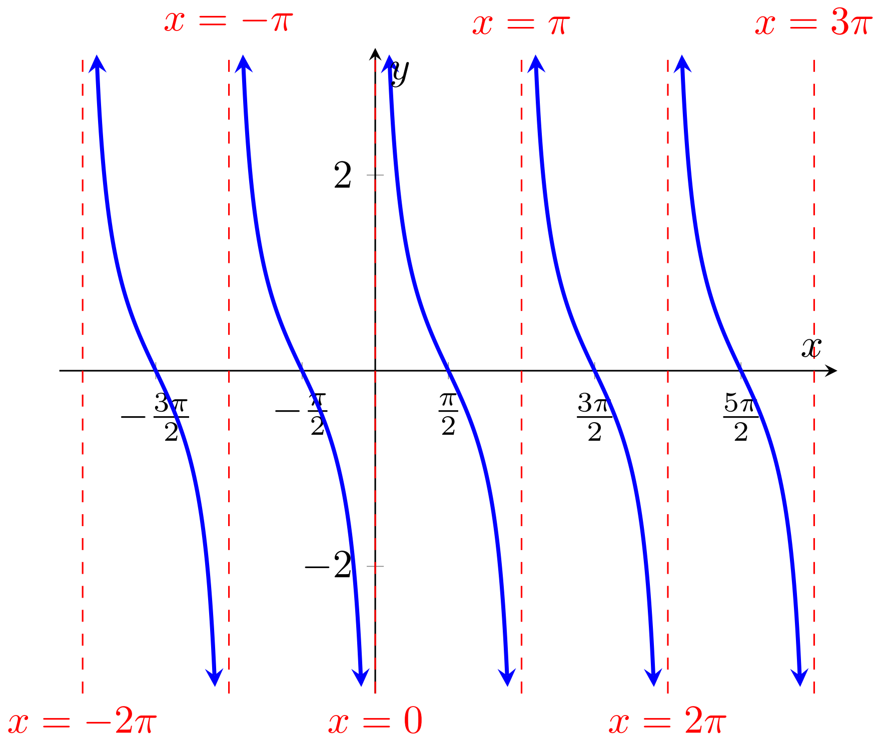

Figure 2.2.14 The graph for $y = \cot(x)$.

The symmetry of the cotangent function about the origin is evident, confirming its nature as an odd function, as explained in Section 1.5.7.

2.2.7 The Cosecant Function

The cosecant function has vertical asymptotes at points $n\pi$, where n is an integer, corresponding to the values where the function is undefined. These points are the same ones excluded from the domain of $\csc x$. As discussed in Subsection 1.5.4, the cosecant function has a period of 2π. Given this, we choose to examine its behavior within one period, specifically from 0 to 2π, since this interval spans one complete period of the cosecant function.

Consider the Reciprocal Identity defining the cosecant function (Definition 1.4.2):

$$\csc x = \frac{1}{\sin x}.$$

As x approaches zero, $\sin x$ decreases to zero, making $\csc x$ approach positive infinity. Increasing x towards $\frac{\pi}{2}$, $\sin(x)$ increases to 1, and cosecant decreases to 1. As x increases from $\frac{\pi}{2}$ to π, $\sin x$ approaches zero, causing $\csc(x)$ to approach infinity.

Similarly, for $x > \pi$ but nearing π, $\sin(x)$ becomes a small, negative number near zero, resulting in $\csc(x)$ approaching negative infinity. As x increases to $\frac{3\pi}{2}$, $\sin(x)$ decreases to -1, and $\csc(x)$ increases to -1. Finally, as x increases from $\frac{3\pi}{2}$ to 2π, $\sin(x)$ approaches zero from the negative side, causing cosecant to approach negative infinity. This behavior is shown in Figure 2.2.15.

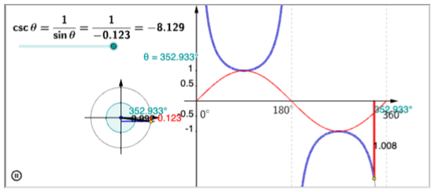

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Figure 2.2.15 As θ moves from $0°$ to $360°$, this figure plots the values of $y = \sin\theta$ and $y = \csc\theta$. Move the slider for θ to see how changing the angle affects $\csc\theta$. Note that while we will generally be using radians when graphing trigonometric functions, this figure uses degrees to help visualize the angle. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the “Standalone” link to explore the interactive version online.

Next we recall values of sine and cosecant for known angles, which are listed in Table 2.2.16 and plotted in Figure 2.2.17.

Table 2.2.16 Values for $y = \csc x$.

x	$\sin x$	$\csc x$	x	$\sin x$	$\csc x$
0	0	Undefined	π	0	Undefined
$\frac{\pi}{6}$	$\frac{1}{2}$	2	$\frac{7\pi}{6}$	$-\frac{1}{2}$	-2
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2}{\sqrt{3}}$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{2}{\sqrt{3}}$
$\frac{\pi}{2}$	1	1	$\frac{3\pi}{2}$	-1	-1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2}{\sqrt{3}}$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{2}{\sqrt{3}}$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	2	$\frac{11\pi}{6}$	$-\frac{1}{2}$	-2
			2π	0	Undefined

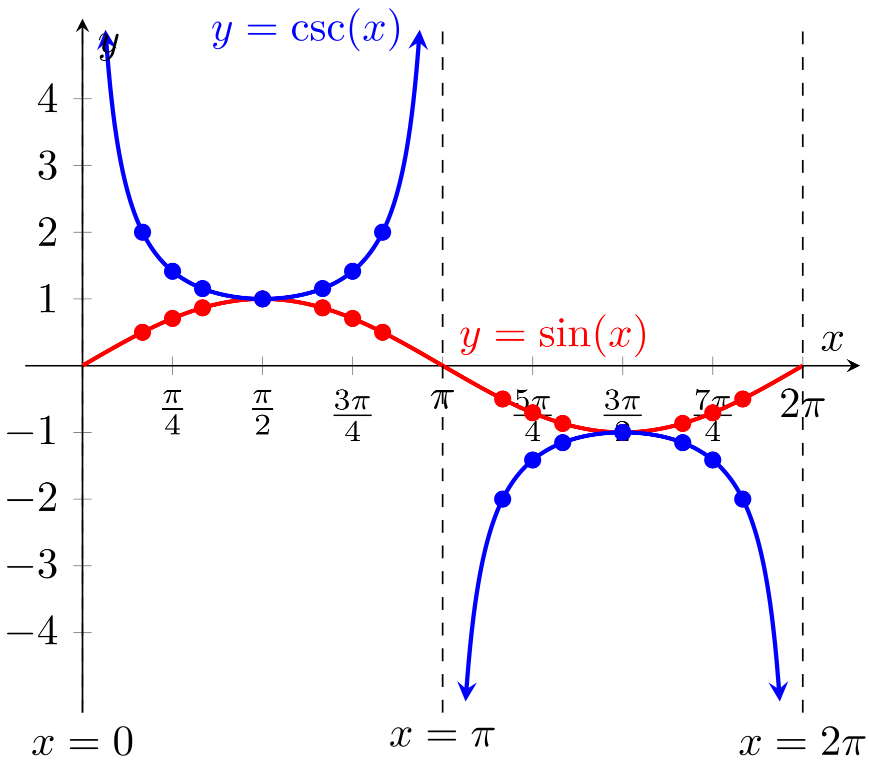

Figure 2.2.17 The values in Table 2.2.16 plotted with a smooth curve connecting the points to make the graph for $y = \csc(x)$.

Since the graph in Figure 2.2.17 represents one period, we can complete the graph of $y = \csc x$ by extending the pattern in both directions to obtain Figure 2.2.18.

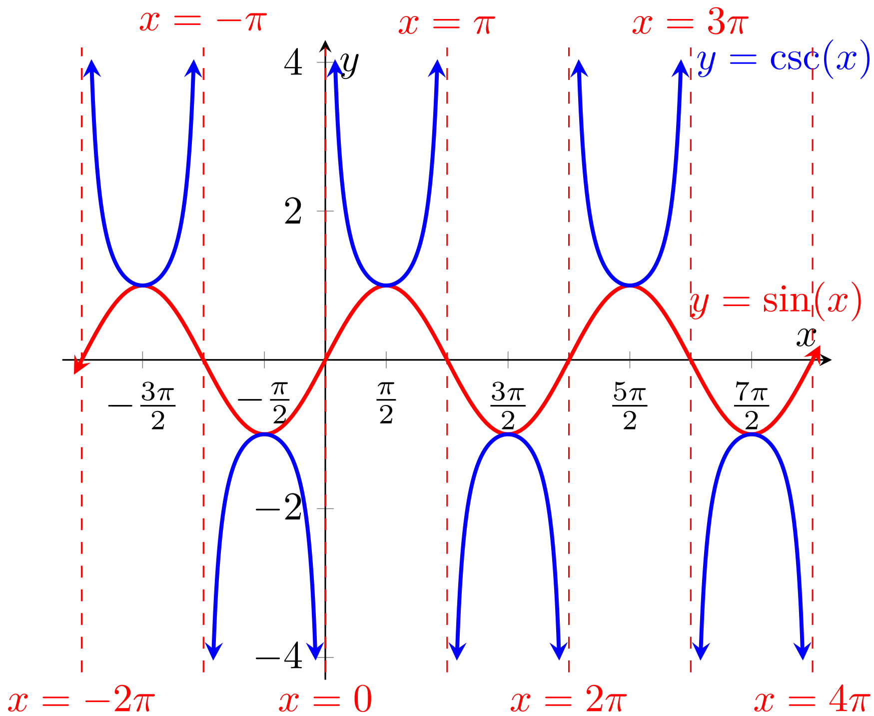

Figure 2.2.18 The graph for $y = \sin(x)$ and $y = \csc(x)$.

Notice the graph of the cosecant function is symmetric with respect to the origin, confirming what we learned in Section 1.5.7, that cosecant is an odd function.

2.2.8 The Secant Function

As discussed earlier in this section, the secant function has a domain of all real numbers except angles of the form $n\frac{\pi}{2}$, where n is an odd integer. These excluded values correspond to the vertical asymptotes of the secant function. With a period of 2π, we can focus on the interval 0 to 2π. The construction of the secant function graph follows a similar approach to the one used for the cosecant function, as illustrated in Figure 2.2.19.

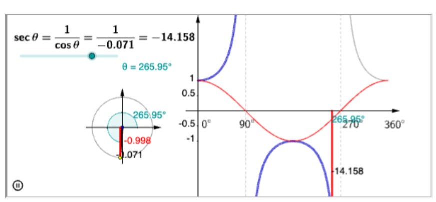

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Figure 2.2.19 As θ moves from 0° to 360°, this figure plots the values of $y = \cos\theta$ and $y = \sec\theta$. Move the slider for θ to see how changing the angle affects $\sec\theta$. Note that while we will generally be using radians when graphing trigonometric functions, this figure uses degrees to help visualize the angle. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

Plotting points for $y = \sec x$ and using the fact that the secant function is periodic, we obtain the graph for secant in Figure 2.2.20.

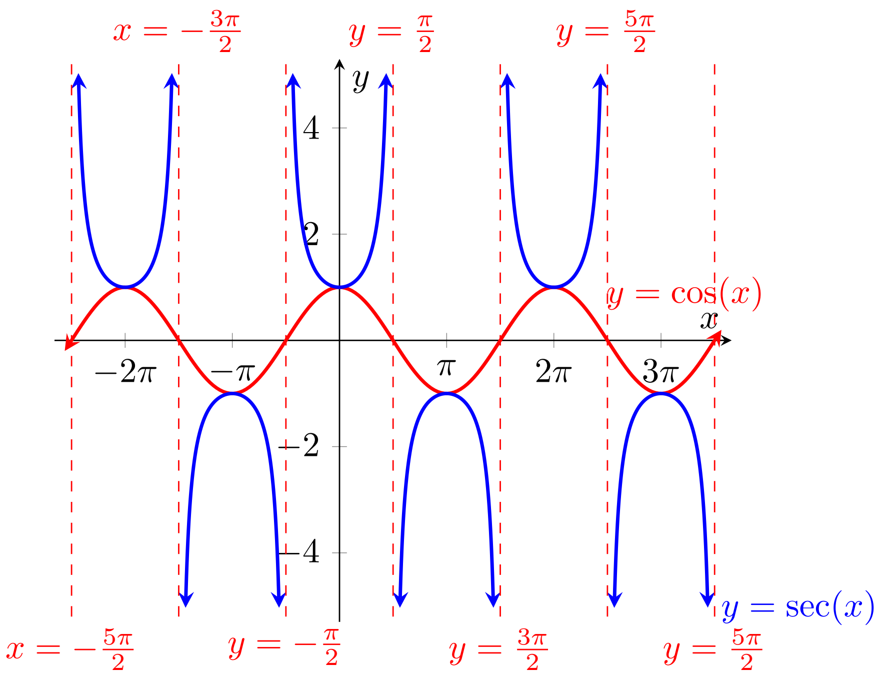

Figure 2.2.20 The graph for $y = \cos(x)$ and $y = \sec(x)$.

Notice the graph of the secant function is symmetric about the y-axis, and thus secant is an even function, confirming what we learned in Section 1.5.7.

2.2.9 Graphing Transformations of Other Trigonometric Functions

Similar to the graphs of sine and cosine, the graphs of the other trigonometric functions can undergo vertical stretching and compressing, horizontal stretching and compressing, phase shifts, vertical shift transformations, and reflections about the x- and y-axes. However, unlike the sine and cosine functions, there is no amplitude for the other trigonometric functions. These transformations are listed in Definition 2.2.21.

Definition 2.2.21 Transformations of the Tangent, Cotangent, Cosecant, and Secant Functions. For functions of the form

$$y = A\tan(B(x - C)) + D, \quad y = A\cot(B(x - C)) + D,$$

$$y = A\csc(B(x - C)) + D, \quad \text{and} \quad y = A\sec(B(x - C)) + D,$$

we can express the transformations as follows:

- Vertical Stretch/Compression: $|A|$
  - $|A|$ is the value of the vertical stretch/compression.
  - If $|A| > 1$, there is vertical stretching.
  - If $0 < |A| < 1$, there is vertical compression.
- Period and Horizontal Stretch/Compression: $|B|$
  - The period is $\frac{\pi}{|B|}$ for tangent and cotangent, and $\frac{2\pi}{|B|}$ for cosecant and secant.

  - If $|B| > 1$, there is horizontal compression, and the period is shortened.
  - If $0 < |B| < 1$, there is horizontal stretching, and the period is lengthened.

- Phase Shift: C
  - If C is positive, there is a shift to the right.
  - If C is negative, there is a shift to the left.

- Vertical Shift: D
  - If D is positive, there is a shift upward.
  - If D is negative, there is a shift downward.

- Reflection about the x-axis:
  - If A is negative ($A < 0$), there is a reflection about the x-axis.

- Reflection about the y-axis:
  - If B is negative ($B < 0$), there is a reflection about the y-axis.

- Vertical Asymptotes:
  - For cotangent and cosecant, vertical asymptotes occur at

    $$x = C + n\frac{\pi}{|B|},$$

    where n is an integer.
  - For tangent and secant, vertical asymptotes occur at

    $$x = C + n\frac{\pi}{2|B|},$$

    where n is an odd integer.

◊

Remark 2.2.22 Other Forms of Transformations. For functions of the form

$$y = A\tan(Bx - E) + D, \quad y = A\cot(Bx - E) + D,$$
$$y = A\csc(Bx - E) + D, \quad \text{and} \quad y = A\sec(Bx - E) + D,$$

the transformations are the same as above, except for the phase shift and vertical asymptotes where we replace C with $\frac{E}{B}$. If $\frac{E}{B} > 0$ the phase shift is to the right, and if $\frac{E}{B} < 0$ it is to the left.

- For cotangent and cosecant, vertical asymptotes occur at

  $$x = \frac{E}{B} + n\frac{\pi}{|B|},$$

  where n is an integer.

- For secant and tangent, vertical asymptotes occur at

  $$x = \frac{E}{B} + n\frac{\pi}{2|B|},$$

where n is an odd integer.

Example 2.2.23 Vertical Stretch/Compression and Reflection about the x-axis. Graph one period for each function.

1. $y = \tan(x)$
2. $y = 2\tan(x)$
3. $y = \dfrac{1}{2}\tan(x)$
4. $y = -\tan(x)$

Solution.

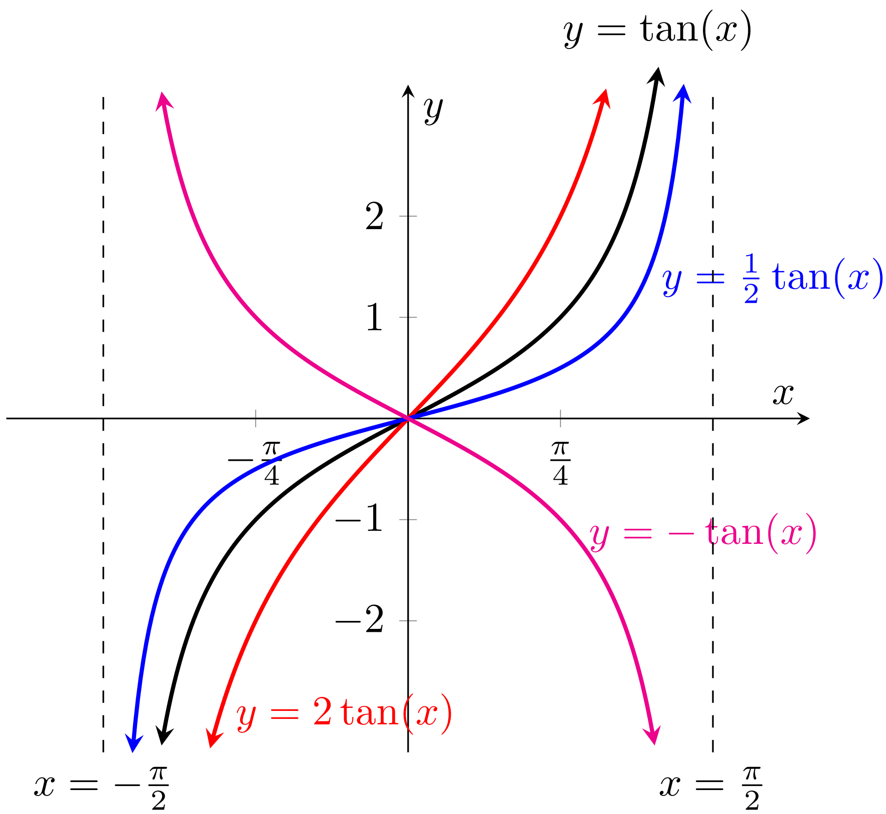

Figure 2.2.24 The transformations of the tangent function graph, starting with the graph of $y = \tan(x)$ along with graphs with a vertical stretch, a vertical compression, and a reflection about the x-axis.

□

Example 2.2.25 Horizontal Stretch/Compression and Reflection about the y-axis. Identify the period and graph one period for each of the following functions:

1. $y = \cot(x)$
2. $y = \cot(2x)$
3. $y = \cot\left(\dfrac{1}{2}x\right)$

4. $y = \cot(-x)$

Solution.

1. The period for $y = \cot(x)$ is π.
2. The period for $y = \cot(2x)$ is $\frac{\pi}{|2|} = \frac{\pi}{2}$.
3. The period for $y = \cot\left(\frac{1}{2}x\right)$ is $\frac{\pi}{\left|\frac{1}{2}\right|} = 2\pi$.
4. The period for $y = \cot(-x)$ is $\frac{\pi}{|-1|} = \pi$.

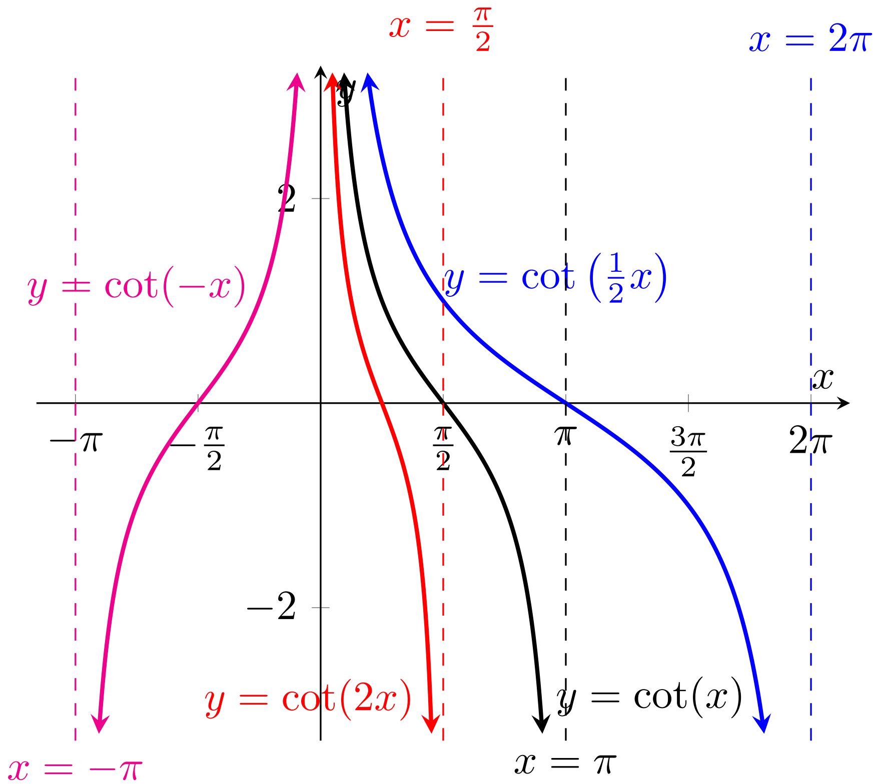

Figure 2.2.26 The transformations of the cotangent function graph, starting with the graph of $y = \cot(x)$ along with graphs with a horizontal stretch, a horizontal compression, and a reflection about the y-axis.

□

Example 2.2.27 Phase Shifts and Vertical Shifts. Graph the function $y = \csc\left(x - \frac{\pi}{2}\right) + 2$.

Solution.

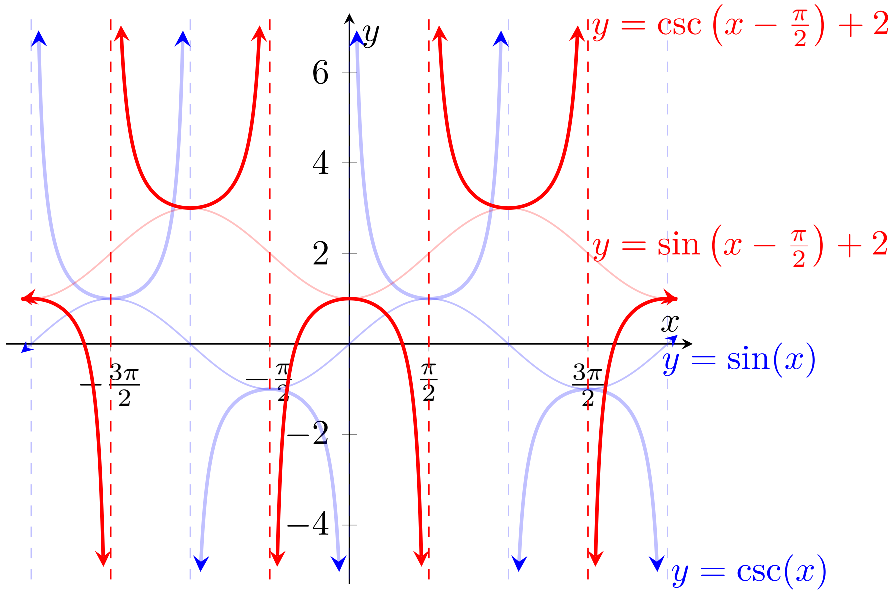

Figure 2.2.28 The graphs of $y = \csc\left(x - \frac{\pi}{2}\right) + 2$ and $y = \sin\left(x - \frac{\pi}{2}\right) + 2$ are derived from the graphs of $y = \csc(x)$ and $y = \sin(x)$ by applying a phase shift of $\frac{\pi}{2}$ to the right and a vertical shift up by 2.

□

Explore the effects of various transformations using the interactive features in Figure 2.2.29 and Figure 2.2.30.

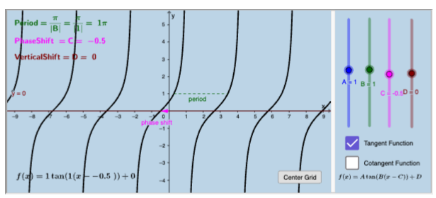

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Figure 2.2.29 Manipulate the graphs of tangent and cotangent by adjusting the sliders for A, B, C, and D. Observe the effects on period, phase and vertical shifts, as well as reflections about the x- and y-axes. Additionally, toggle between the tangent and cotangent graphs by selecting the corresponding function. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

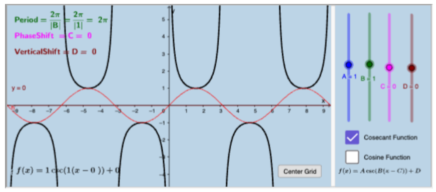

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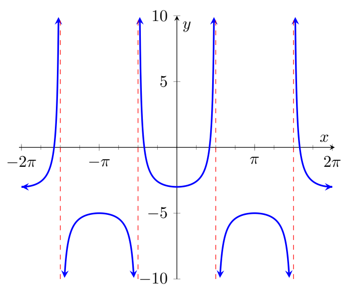
Figure 2.2.30 Manipulate the graphs of cosecant and secant by adjusting the sliders for A, B, C, and D. Observe the effects on period, phase and vertical shifts, as well as reflections about the x- and y-axes. Additionally, toggle between the cosecant and secant graphs by selecting the corresponding function. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

2.2.10 Exercises

Exercise Group. Graph the function.

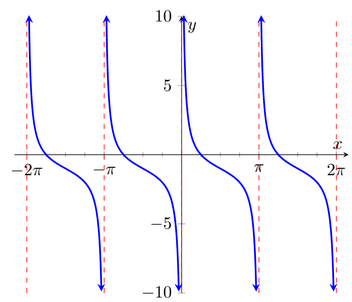
1. $y = \sec(x) - 4$

Answer.



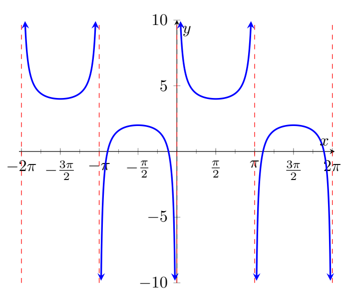
2. $y = \cot(x) - 1$

Answer.



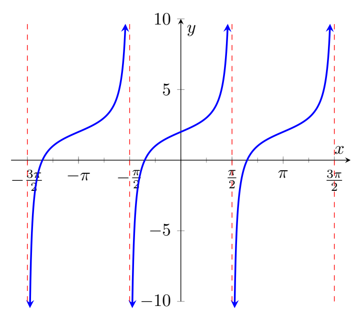
3. $y = \csc(x) + 3$

Answer.



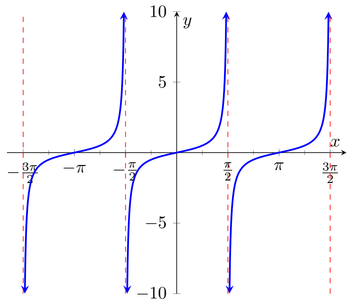
4. $y = \tan(x) + 2$

Answer.



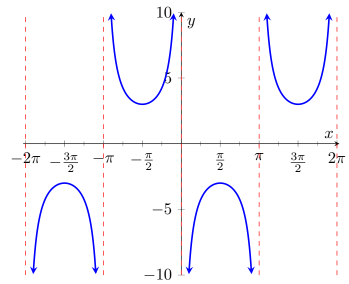
5. $y = \frac{1}{2}\tan(x)$

Answer.



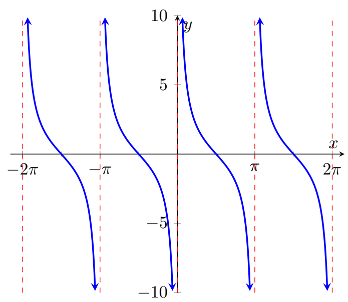
6. $y = -3\csc(x)$

Answer.



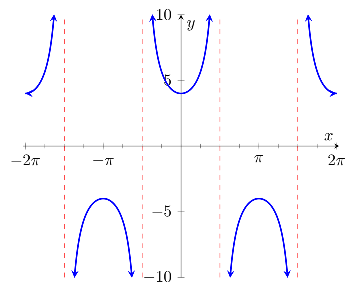
7. $y = 2\cot(x)$

Answer.



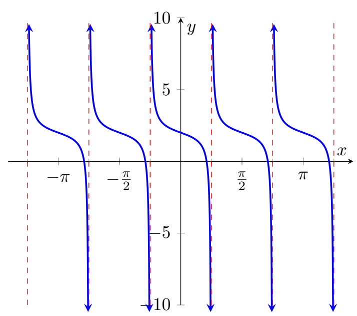
8. $y = 4\sec(x)$

Answer.



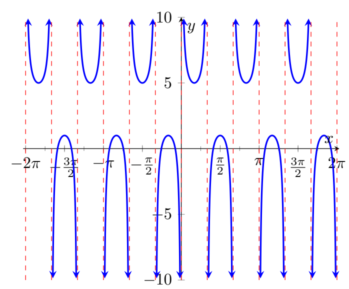
9. $y = -\frac{1}{2}\tan(2x) + 2$

Answer.



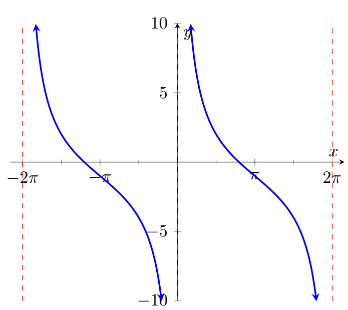
10. $y = 2\csc(3x) + 3$

Answer.



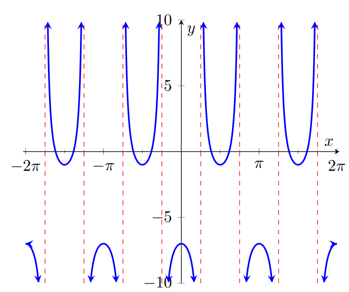
11. $y = 3\cot\left(\frac{1}{2}x\right) - 1$

Answer.



12. $y = -3\sec(2x) - 4$

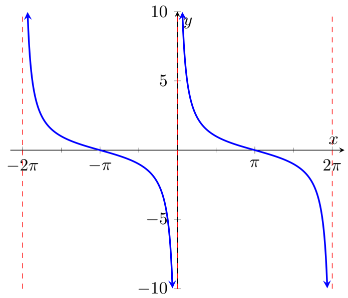
Answer.



Exercise Group. For each function, determine the period and sketch the graph.

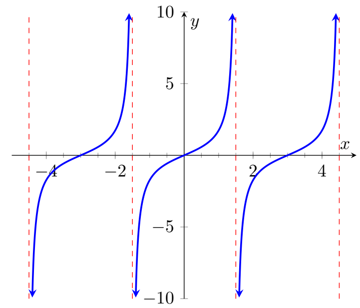
13. $y = \cot\left(\frac{1}{2}x\right)$

Answer. Period: 2π



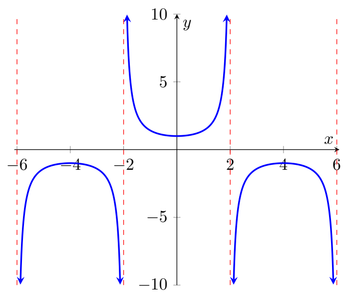
14. $y = \tan\left(\frac{\pi}{3}x\right)$

Answer. Period: 3



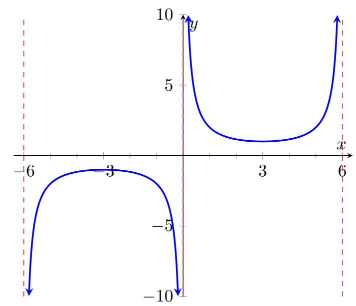
15. $y = \sec\left(\frac{\pi}{4}x\right)$

Answer. Period: 8



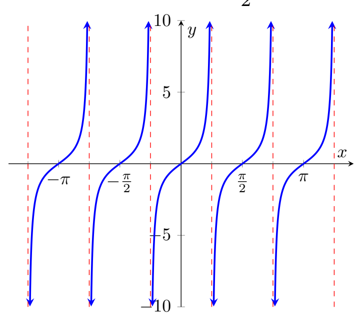
16. $y = \csc\left(\frac{\pi}{6}x\right)$

Answer. Period: 12



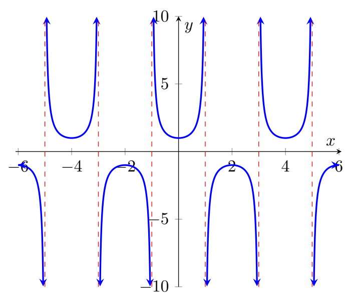
17. $y = \tan(2x)$

Answer. Period: $\frac{\pi}{2}$



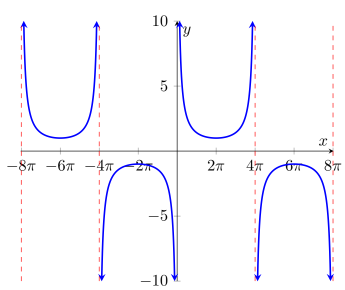
18. $y = \sec\left(\frac{\pi}{2}x\right)$

Answer. Period: 4



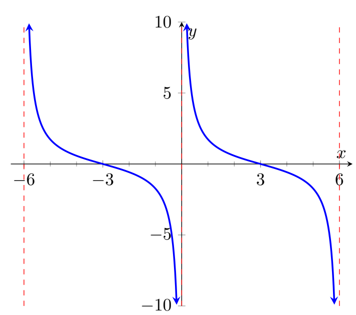
19. $y = \csc\left(\frac{1}{4}x\right)$

Answer. Period: 8π

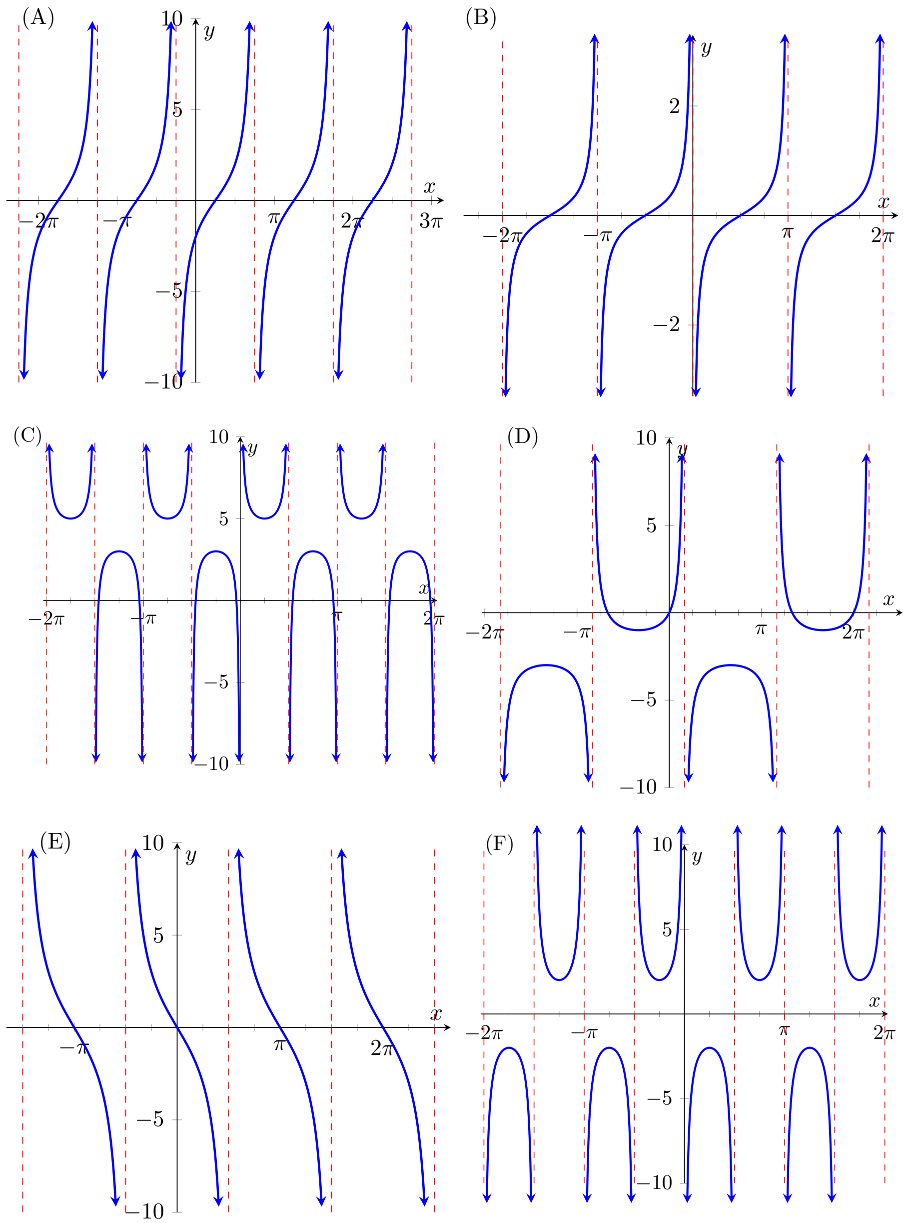


20. $y = \cot\left(\frac{\pi}{6}x\right)$

Answer. Period: 6



Exercise Group. Match each given function to one of the graphs below.



21. $y = \csc(2x) + 4$

Answer. C

22. $y = 2\tan\left(x - \frac{\pi}{4}\right)$

Answer. A

23. $y = 3\cot\left(x - \frac{\pi}{2}\right)$

Answer. E

24. $y = -\frac{1}{3}\cot(x)$

Answer. B

25. $y = -2\csc(2x)$

Answer. F

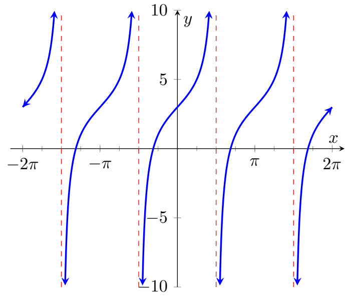
26. $y = \sec\left(x + \frac{\pi}{3}\right) - 2$

Answer. D

Exercise Group. Find the period, phase shift, and vertical shift of each function and sketch the graph.

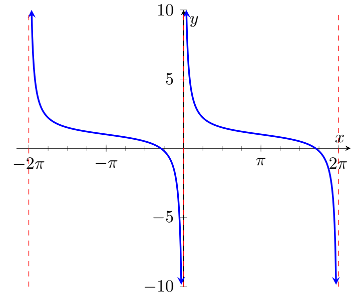
27. $y = 2\tan(x) + 3$

Answer. Period: π; Phase Shift: 0; Vertical Shift: 3



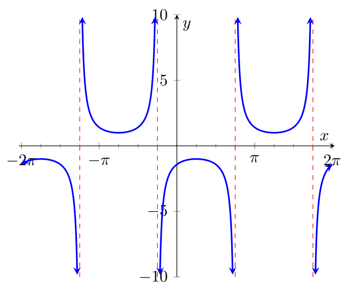
28. $y = \frac{1}{2}\cot\left(\frac{1}{2}x\right) + 1$

Answer. Period: 2π; Phase Shift: 0; Vertical Shift: 1



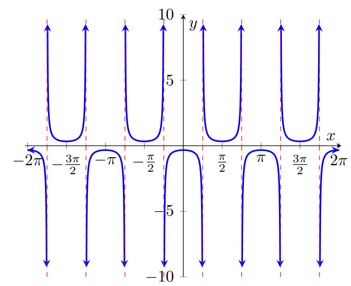
29. $y = -\csc\left(x + \frac{\pi}{4}\right)$

Answer. Period: 2π; Phase Shift: $-\frac{\pi}{4}$; Vertical Shift: 0



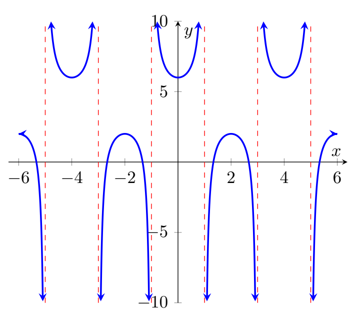
30. $y = -\frac{1}{3}\sec(2x)$

Answer. Period: π; Phase Shift: 0; Vertical Shift: 0



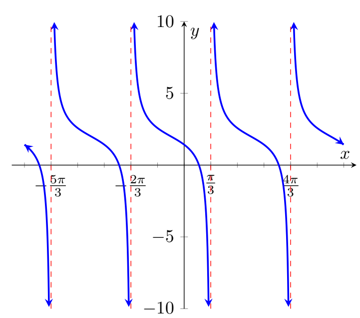
31. $y = 2\sec\left(\frac{\pi}{2}x\right) + 4$

Answer. Period: 4; Phase Shift: 0; Vertical Shift: 4



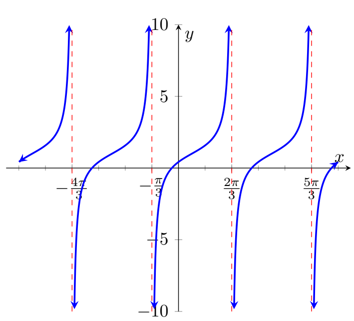
32. $y = \cot\left(x - \frac{\pi}{3}\right) + 2$

Answer. Period: π; Phase Shift: $\frac{\pi}{3}$; Vertical Shift: 2



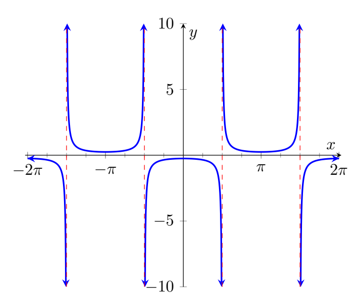
33. $y = \tan\left(x - \frac{\pi}{6}\right) + 1$

Answer. Period: π; Phase Shift: $\frac{\pi}{6}$; Vertical Shift: 1



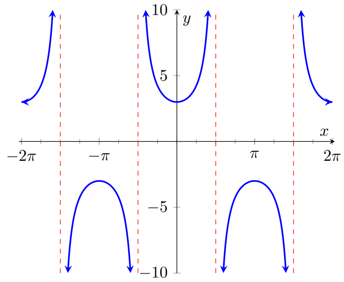
34. $y = -\frac{1}{4}\sec(x)$

Answer. Period: 2π; Phase Shift: 0; Vertical Shift: 0



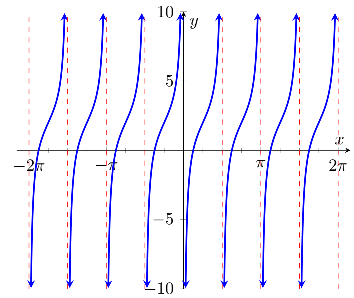
35. $y = 3\csc\left(x + \frac{\pi}{2}\right)$

Answer. Period: 2π; Phase Shift: $-\frac{\pi}{2}$; Vertical Shift: 0



36. $y = -\tan\left(2\left(x - \frac{\pi}{4}\right)\right) + 2$

Answer. Period: $\frac{\pi}{2}$; Phase Shift: $\frac{\pi}{4}$; Vertical Shift: 2



Navigating with Shadows. When navigating the open ocean, maintaining a straight course poses challenges due to limited visual markers. One technique involves the steersperson using the positions of shadows cast by objects on the canoe—such as crew members, railings, and sails—to keep them fixed on the deck, ensuring a straight trajectory. However, if the canoe veers off course, the changing position of the canoe relative to the sun leads to a shift in the shadows. Observing these shadow movements allows the steersperson to make course corrections. It's important to note that this method is effective only over a short duration, as the sun's continuous movement across the sky causes ongoing changes in shadow positions. To illustrate the limitations over extended periods, consider the example of the Samoan double-hulled voyaging vaʻa, Gaualofa, with a 14-meter-high mast. The length of the shadow is modeled by

$$l(t) = 14\left|\cot\left(\frac{\pi}{12}t\right)\right|,$$

where l is the shadow length in meters and t represents the hours since 6 am (assuming sunrise at 6 am and sunset at 6 pm). In each of the following questions, calculate the length of the shadow, rounded to the nearest tenth of a meter, for the given time.

37. 7:00am

Answer. 52.2 meters

38. 10:00am

Answer. 8.1 meters

39. 12:00pm

Answer. 0 meters

40. 3:00pm

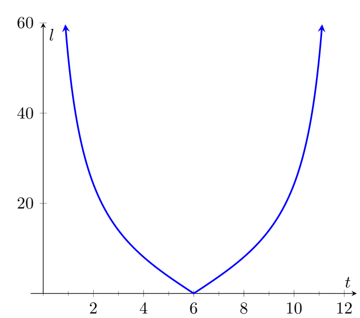
Answer. 14 meters

41. 4:00pm

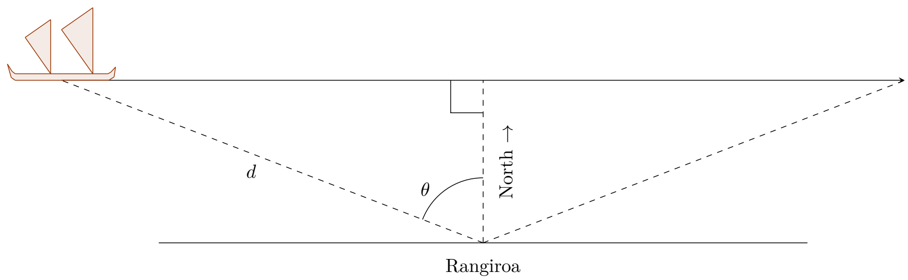
Answer. 24.2 meters

42. Graph the length of the shadow, l, throughout the day from 6:00am to 6:00pm $(0 < t < 12)$

Answer.



Exercise Group.



An observer on Rangiroa spots the Faʻafaite, a double-hulled voyaging canoe from Tahiti, sailing off the north coast of the atoll, maintaining a distance of 1.5 kilometers from the shore and traveling east. Let θ represent the angle formed between the line from the observer to the vaʻa and a line extending due north from the observer, measured in radians. The angle θ is negative if the vaʻa is to the left of the observer and positive when to the right, as shown in the figure above. The distance (in kilometers), denoted by $d(\theta)$ from Faʻafaite to the observer is given by the function

$$d(\theta) = 1.5\sec(\theta).$$

In each of the following questions, calculate the distance from the observer to Faʻafaite, $d(\theta)$, in kilometers, for the given angle θ. Round your answer to two decimal places.

43. $\theta = -\frac{\pi}{3}$

Answer. 3.00 km

44. $\theta = -\frac{\pi}{4}$

Answer. 2.12 km

45. $\theta = -\frac{\pi}{6}$

Answer. 1.73 km

46. $\theta = 0$

Answer. 1.50 km

47. $\theta = \frac{\pi}{6}$

Answer. 1.73 km

48. $\theta = \frac{\pi}{4}$

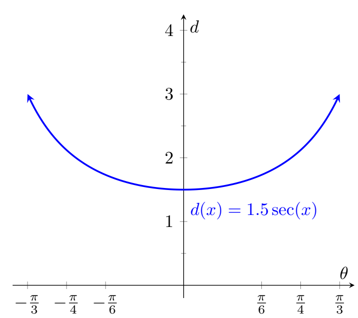
Answer. 2.12 km

49. $\theta = \frac{\pi}{3}$

Answer. 3.00 km

50. Graph the function $d(\theta)$ on the domain $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$.

Answer.



51. What happens to d as θ approaches $\frac{\pi}{2}$?

Answer. As θ approaches $\frac{\pi}{2}$, the function $d(\theta) = 1.5\sec(\theta)$ approaches positive infinity.

52. What is the closest distance Faʻafaite comes to shore? Where does this occur?

Answer. 1.5 km, when Faʻafaite is directly north of the observer ($\theta = 0$).

2.3 Sinusoidal Curve Fitting and Graphical Analysis

In this chapter, we learn that sinusoidal patterns exist in various aspects throughout our world. One example is the moon, which undergoes phases oscillating from no illumination, to waxing (increasing illumination), reaching a fully lit moon, and then waning (decreasing illumination) until it completes its cycle with no illumination again. Indigenous cultures across the world have deeply connected with these lunar cycles, shaping cultural practices aligned with the moon's phases. For example, the Māori of New Zealand and the Hopi Tribe in northeastern Arizona, USA, both time activities such as planting and harvesting specific plants for each moon cycle based on generations of lunar observations.

In Section 2.1, we graphed sinusoidal functions and determined their values for any given input value. The ability to formulate a sinusoidal equation modeling the moon's phases allows us to predict the moon's phase on any date. In this section, we will explore the process of developing sinusoidal functions based on provided information. This will enable us to model real-world phenomena using real data.

2.3.1 Finding Sinusoidal Equations from Characteristics

To begin finding sinusoidal equations of the form

$$y = A\sin(B(x-C)) + D \quad \text{and} \quad y = A\cos(B(x-C)) + D$$

we will use the characteristics of sine and cosine functions, as described in Remark 2.1.37 and summarized below, to determine the values of A, B, C, and D.

- Amplitude and Vertical Stretch/Compression: $|A|$
  - $|A|$ is the value of the amplitude.
  - If $|A| > 1$, there is vertical stretching.
  - If $0 < |A| < 1$, there is vertical compression.
- Period and Horizontal Stretch/Compression: $|B|$
  - The period is $\frac{2\pi}{|B|}$.
  - If $|B| > 1$, there is horizontal compression and the period is shortened.
  - If $0 < |B| < 1$, there is horizontal stretching and the period is lengthened.
- Phase Shift: C
  - If C is positive, there is a shift to the right.
  - If C is negative, there is a shift to the left.
- Vertical Shift: D
  - If D is positive, there is a shift upward.
  - If D is negative, there is a shift downward.

- Reflection about the x-axis:
  - If A is negative ($A < 0$), there is a reflection about the x-axis.
- Reflection about the y-axis:
  - If B is negative ($B < 0$), there is a reflection about the y-axis.

Example 2.3.1 Finding an Equation of a Sine Function Using its Characteristics. Write an equation of a sine function with an amplitude of 2, period of 8, phase shift of $\frac{\pi}{3}$, and vertical shift of 4.

Solution. To determine an equation of the form

$$f(x) = A\sin(B(x - C)) + D$$

we first examine the information that we are given: amplitude of 2, period of 8, phase shift of $\frac{\pi}{3}$, and vertical shift of 4.

Since $|A|$ represents the value of the amplitude, we get

$$|A| = 2.$$

Thus we either have $A = 2$ or $A = -2$. Since we are not given a reflection about the x-axis, we can conclude that A is not negative, thus

$$A = 2.$$

Next, since the period of a sine function is given by $\frac{2\pi}{|B|}$, we get

$$\frac{2\pi}{|B|} = 8.$$

Solving for $|B|$, we get

$$|B| = \frac{2\pi}{8} = \frac{\pi}{4}.$$

Since there is no reflection about the y-axis, we have that B must be positive:

$$B = \frac{\pi}{4}.$$

Finally, since C and D represent the phase shift and vertical shift, respectively, we get

$$C = \frac{\pi}{3}, \quad D = 4.$$

Combining these, the sine function becomes:

$$f(x) = 2\sin\left(\frac{\pi}{4}\left(x - \frac{\pi}{3}\right)\right) + 4.$$

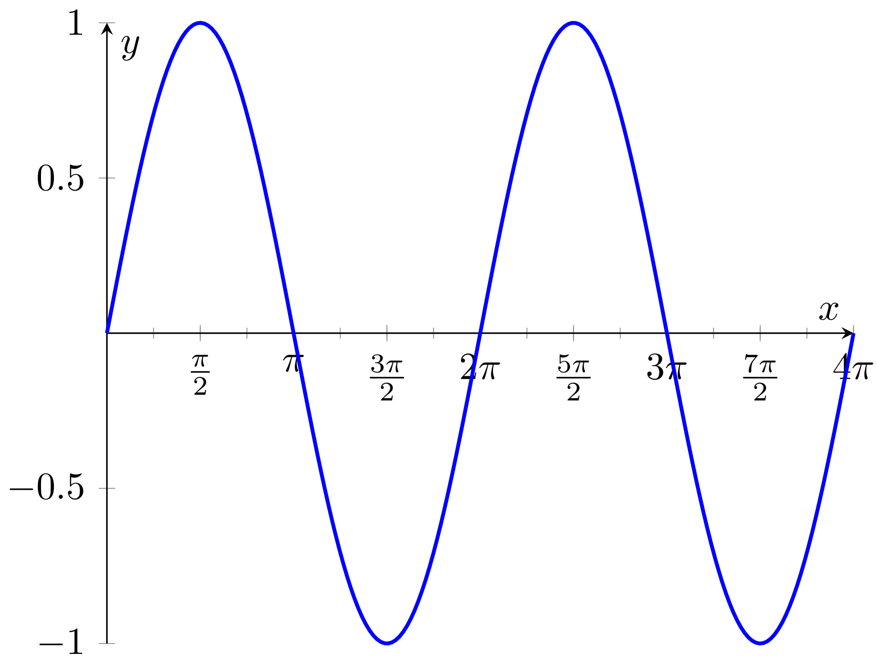
□

2.3.2 Finding Sinusoidal Equations from Graphs

Sometimes we are not explicitly given the characteristics of the function, but are provided with the graph. Examining a graph can reveal its characteristics, allowing us to find the equation of a function.

In this next example, we'll explore how to find an equation of a cosine function based on the graph of a sine function.

Example 2.3.2 Finding an Equation of a Cosine Function Using the Graph of a Sine Function. Below is the graph of $y = \sin(x)$.



Find an equation of the form $y = A\cos(B(x - C)) + D$ that fits the graph.

Solution. The cosine function has its maximum when $x = 0$, decreases to its minimum at $x = \pi$, and then increases before completing one period at 2π. This graph starts at 0 when $x = 0$, then increases to its maximum at $x = \frac{\pi}{2}$, and then has the shape of the cosine function starting at $x = 0$. This represents a phase shift to the right. There no reflections about the x-axis or y-axis. We will find the equation of a function with characteristics of a phase shift to the right.

Amplitude (A): Since the amplitude of the cosine function will be the same as the amplitude of the sine function, which is 1, we have $|A| = 1$. Since there is no reflection about the x-axis, we choose the positive value to get $A = 1$.

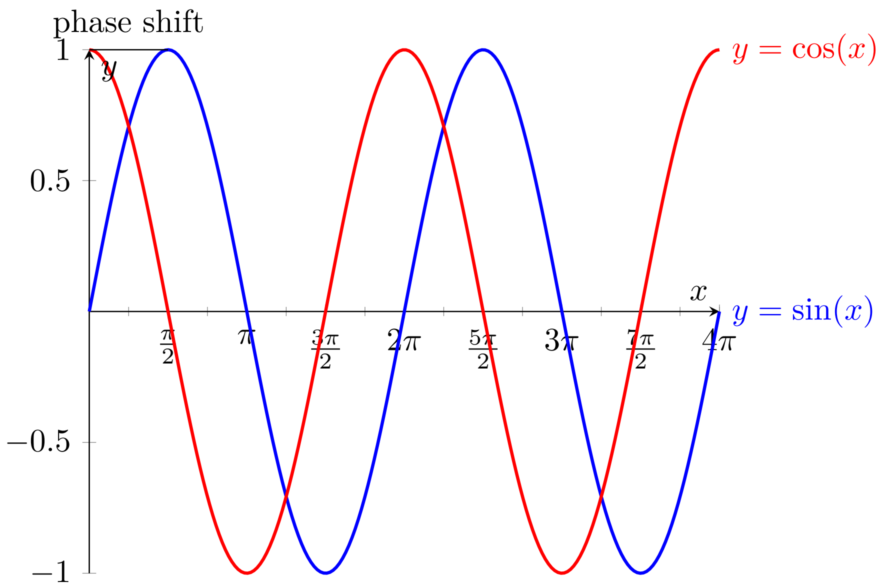
Vertical Shift (D): The vertical shift of the cosine function will be the same as that of the sine function, which is 0. So, we have $D = 0$.

Period (B): The period of the cosine function will also be the same as the period of the sine function, which is 2π. Since $2\pi = \frac{2\pi}{|B|}$, we have $|B| = 1$. Since there is no reflection about the y-axis, we get $B = 1$.

Therefore, we have

$$y = 1\cos(1x) + 0 = \cos(x).$$

Overlapping the graph of $y = \cos(x)$ onto the original graph will help us determine the phase shift.



Phase Shift (C): From the graph, the cosine function will have a phase shift of $\frac{\pi}{2}$ radians to the right to match the graph of the sine function. This is because the cosine function reaches its maximum value at $x = 0$, while the sine function reaches its maximum at $x = \frac{\pi}{2}$. So, we have $C = \frac{\pi}{2}$.

Therefore, the equation of the cosine function that fits the graph of $y = \sin(x)$ is:

$$y = \cos\left(x - \frac{\pi}{2}\right)$$

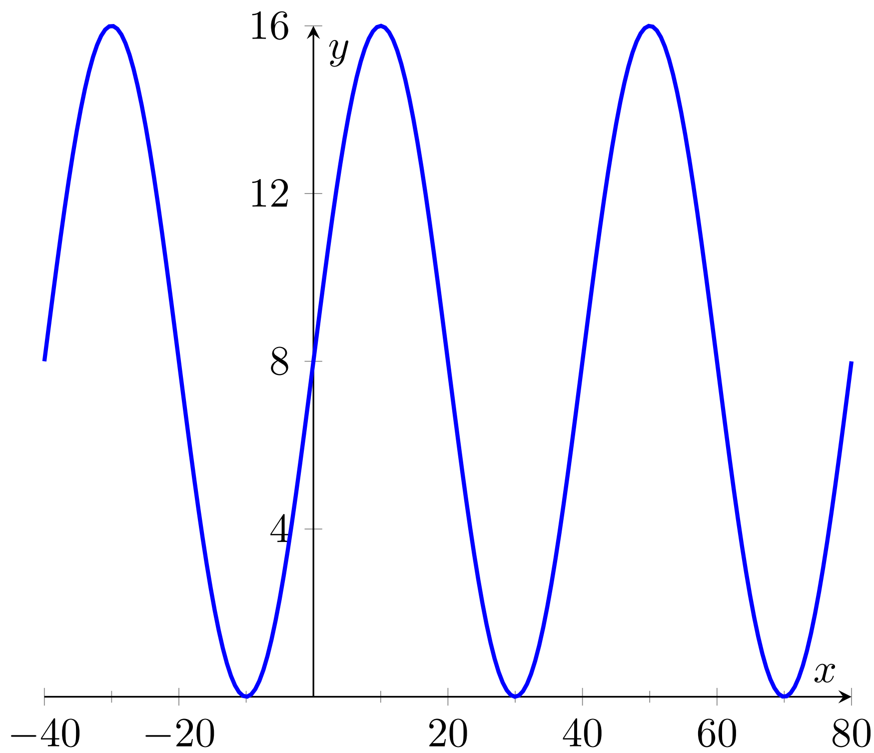
□

Remark 2.3.3 Since this graph is the graph of $y = \sin(x)$, we get

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right).$$

This solution demonstrates that any sine function can be written as a cosine function with an appropriate phase shift. In this case, the phase shift of $\frac{\pi}{2}$ radians to the right transforms the cosine function into its corresponding sine function.

Example 2.3.4 Finding an Equation of a Sine Function Using its Graphs. Given the graph below, find an equation that represents the graph in the following forms:



(a)

$$f(x) = A\sin(B(x - C)) + D.$$

Solution. The sine function starts at the origin, increases to its maximum when $x = \frac{\pi}{2}$, decreases to its minimum at $x = \frac{3\pi}{2}$, and then increases again before completing one period at 2π. In the given graph, the function has a value of 8 when $x = 0$, then increases to its maximum value of 16 at $x = 10$, decreases to its minimum value of 0 at $x = 30$, and then increases again before completing one period at $x = 40$. This indicates a vertical shift, a vertical stretch, and a horizontal stretch of the period. There are no reflections about the x-axis or y-axis. Therefore, we need to find the equation of a function with these characteristics."

Amplitude (A): Recall from Definition 2.1.23 that the amplitude is defined as half of the difference between the maximum and minimum values of the function. Here, the maximum value is 16 and the minimum value is 0, so the amplitude is:

$$|A| = \frac{16 - 0}{2} = 8.$$

Since there are no reflections about the x-axis, we use the positive value to get $A = 8$.

Vertical Shift (D): The midline, calculated as

$$y = \frac{\text{maximum} + \text{minimum}}{2} = \frac{16 + 0}{2} = 8,$$

represents the vertical shift of the graph. Therefore,

$$D = 8.$$

Period (B): The period of a function is the length of one full repetition (or cycle). We can identify the period on the graph by measuring the horizontal distance between corresponding points where the graph completes a cycle. In this case, we'll use two corresponding peaks at $x = 10$ and $x = 50$ to obtain:

$$\text{period } = 50 - 10 = 40.$$

Note that other points on the graph, such as the minimum values or the points where the graph crosses the midline, could also be used to determine the period.

Since Definition 2.1.27 defines a period as $\frac{2\pi}{|B|}$, we have:

$$40 = \frac{2\pi}{|B|}.$$

Thus,

$$|B| = \frac{2\pi}{40} = \frac{\pi}{20}.$$

Since there are no reflections about the y-axis, we get $B = \frac{\pi}{20}$.

Phase Shift (C): The phase shift of the graph refers to its horizontal translation. The cycle of a sine function typically starts at $x = 0$ on the midline, increases to the maximum, decreases, passes the midline to the minimum, and then completes a cycle back at the midline. Since this graph follows these characteristics without any horizontal translation, there is no phase shift. Therefore, $C = 0$.

Thus, the graph can be described by the following sine function:

$$f(x) = 8\sin\left(\frac{\pi}{20}x\right) + 8.$$

(b)

$$g(x) = A\cos(B(x - C)) + D.$$

Solution. Since the cosine function has its maximum value when $x = 0$ before decreasing to its minimum and then increasing to complete one period, the graph provides several options for phase shifts. By selecting a peak and determining the direction and value of the phase shift needed for the cosine function to reach that peak, we can align it with the given graph. For example, if we choose the peak at $x = -30$, the cosine function will shift 30 units to the left. Similarly, for peaks at $x = 10$ or $x = 50$, the cosine function would need to shift 10 or 50 units, respectively, to the right. Since the peak at $x = 10$ is the closest to the peak of the original cosine function when $x = 0$, we opt for this phase shift. Additionally, as the graph of the sine function, there is a vertical shift, a vertical stretch, and a horizontal stretch, with no reflections about the y-axis or the x-axis.

Amplitude (A): The amplitude of a cosine function is the same as that of the corresponding sine function. Thus, $|A| = 8$. Since there are no reflections about the x-axis, we use the positive value to get $A = 8$.

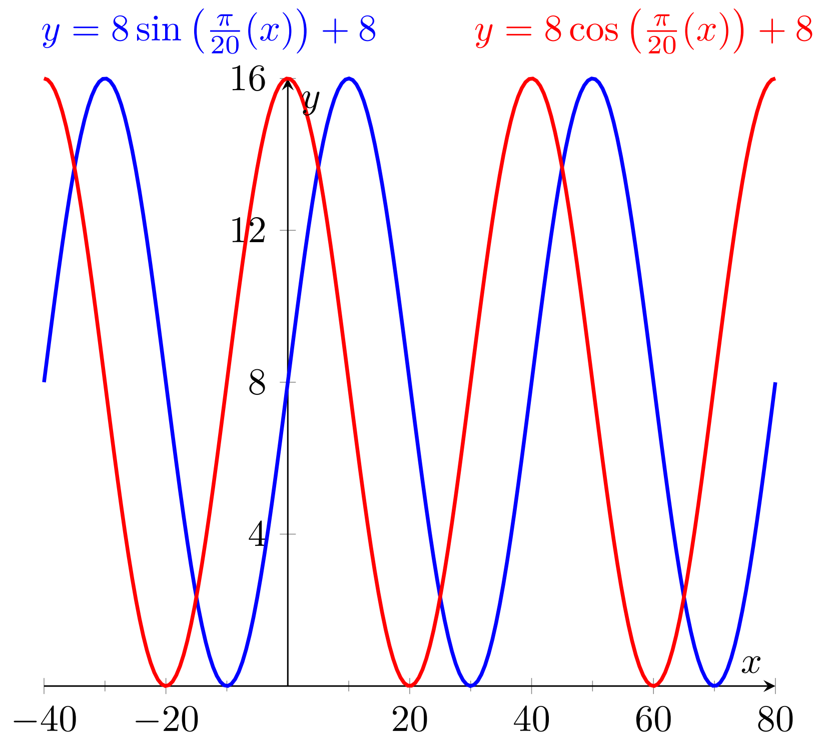
Vertical Shift (D): The vertical shift of a cosine function is the same as that of the corresponding sine function. Thus, $D = 8$.

Period (B): The period of a cosine function is also the same as that of the corresponding sine function. In this case, $|B| = \frac{\pi}{20}$. Since there are no reflections about the y-axis, we determine $B = \frac{\pi}{20}$.

Before we examine the phase shift, let's summarize what we found so far:

$$g(x) = 8\cos\left(\frac{\pi}{20}(x)\right) + 8$$

and overlap this graph with the given graph.



Phase Shift (C): From the graph, we see that the starting point of the cosine function's cycle is at its maximum value, unlike the starting point of a sine function's cycle, which is at its midline. In this case, the graph has a horizontal shift to the right of 10 units. Thus, $C = 10$.

We can now describe the graph with the following cosine function:

$$g(x) = 8\cos\left(\frac{\pi}{20}(x - 10)\right) + 8.$$

□

2.3.3 Finding Sinusoidal Equations from Data

Example 2.3.5 Modeling the Daylight Hours in Munda. In Section 2.1 we learned that Earth's axis is tilted, and as the Earth orbits the sun, this axial tilt causes seasons, which are periodic. Another effect of the axial tilt and orbit is the amount of daylight each part of Earth experiences, which is also periodic.

The total sunlight duration in Munda, on the island of New Georgia in the Solomon Islands in 2025, is plotted in Figure 2.3.6. On the 21st of June 2025 (the 172nd day of the year), which is the shortest day of the year in Munda, the total sunlight duration is 11 hours, 38 minutes, and 17 seconds (approximately 11.64 hours). Conversely, on the 22nd of December 2025 (the 356th day of the year), which is the longest day of the year, the total sunlight duration is 12 hours, 36 minutes, and 44 seconds (approximately 12.61 hours). Find a function of the form $y = A\cos(B(x - C)) + D$ to model the total hours of daylight in

Munda in 2025, assuming that one period represents one year or 365 days.

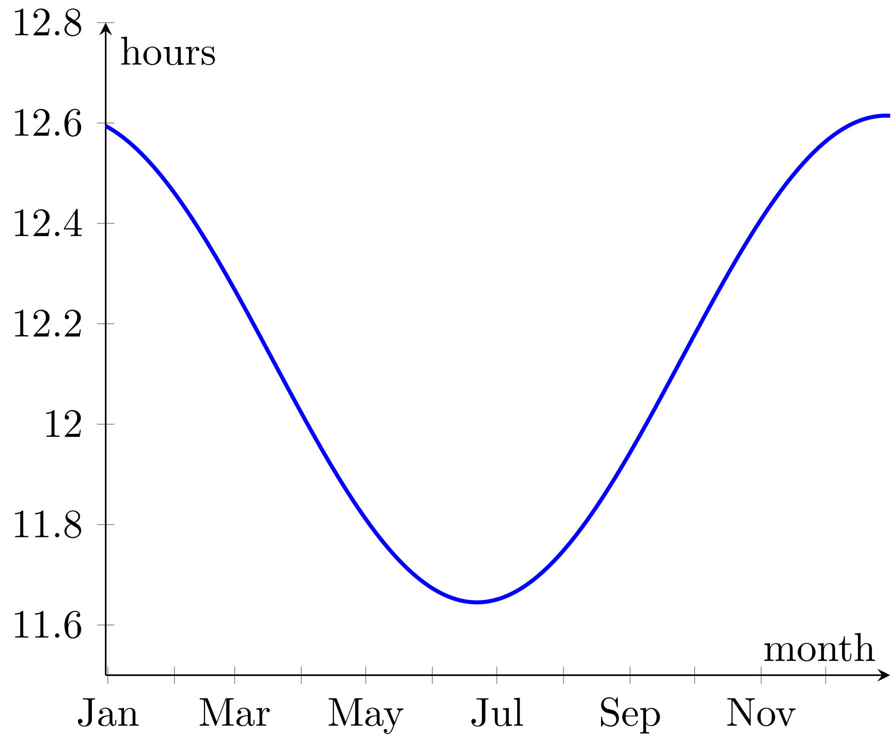


Figure 2.3.6 Hours of daylight in Munda, Solomon Islands. Source: NOAA Solar Calculator.

Solution. *Amplitude (A)*: From the definition of amplitude (Definition 2.1.23),

$$|A| = \frac{\text{maximum} - \text{minimum}}{2} = \frac{12.61 - 11.64}{2} = 0.485.$$

Assuming no reflections about the x-axis, we have:

$$A = 0.485.$$

Period (B): We assume a period of one year or 365 days. Additionally, we can assume no reflections about the y-axis, leading to a positive value for B. Thus we have $\frac{2\pi}{B} = 365$ or

$$B = \frac{2\pi}{365} \approx 0.0172.$$

Phase Shift (C): The characteristic of a cosine function is that at $x = 0$, the function is at its maximum value. However, in this case, the maximum value occurs on Day 356. This represents a phase shift to the right by 356 days, thus:

$$C = 356.$$

Vertical Shift (D): The value of the midline represents the average duration of daylight, which is the vertical shift:

$$D = \frac{\text{maximum} + \text{minimum}}{2} = \frac{12.61 + 11.64}{2} = 12.125.$$

Therefore, the equation of our function is

$$y = 0.485 \cos(0.0172(x - 356)) + 12.125.$$

□

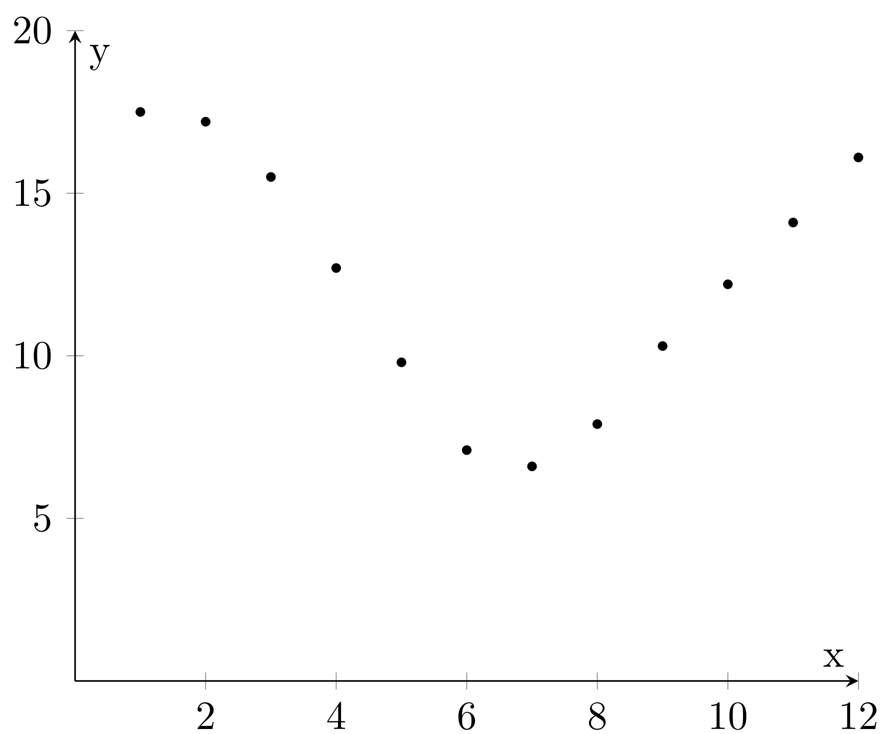
Remark 2.3.7 It's important to note that the average duration of daylight is not exactly 12 hours but instead approximately 12.125 hours. This discrepancy arises due to atmospheric refractions, which cause the apparent sunrise and sunset to occur slightly before and after, respectively, the sun crosses the horizon—the actual sunrise and sunset times.

Example 2.3.8 Modeling the Temperature in Christchurch. Another effect of axial tilt besides daylight hours is temperature, which is also periodic. The average monthly temperature for Christchurch, New Zealand is given in Table 2.3.9 (Source: National Institute of Water and Atmospheric Research (NIWA). Retrieved 18 March 2024). Find a sinusoidal function of the form $y = A\cos(B(x-C))+D$ to model the average monthly temperature of Christchurch.

Table 2.3.9 The average monthly temperature for Christchurch, New Zealand.

Month, x	Temperature (°C)
January, 1	17.5
February, 2	17.2
March, 3	15.5
April, 4	12.7
May, 5	9.8
June, 6	7.1
July, 7	6.6
August, 8	7.9
September, 9	10.3
October, 10	12.2
November, 11	14.1
December, 12	16.1

Solution. We begin by plotting the points in Table 2.3.9.



Examining the plot of the data points, we see this looks like the graph of a cosine function with no reflections about the x-axis or the y-axis.

Amplitude (A): The amplitude is

$$|A| = \frac{\text{maximum} - \text{minimum}}{2} = \frac{17.5 - 6.6}{2} = 5.45.$$

Since there are no reflections about the x-axis, we get $A = 5.45$

Period (B): Since the temperatures repeat every 12 months, the period is 12 and so $\frac{2\pi}{|B|} = 12$. Since there are no reflections about the y-axis, we keep the positive value to obtain

$$B = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Vertical Shift (D): The vertical shift is the average of the maximum and minimum values of the data, which corresponds to the midline of the graph:

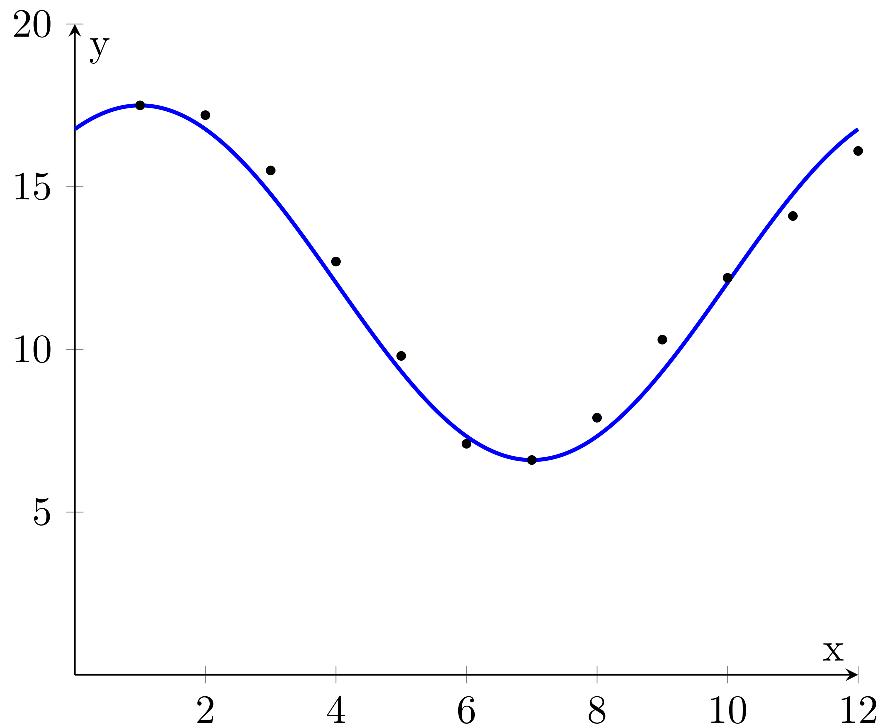
$$D = \frac{\text{maximum} + \text{minimum}}{2} = \frac{17.5 + 6.6}{2} = 12.05.$$

Phase Shift (C): The maximum temperature in the data occurs in January ($x = 1$). However, since the graph of cosine reaches its maximum value at $x = 0$, we have a phase shift of 1 to the right to align the peak with the maximum temperature data point. Thus, $C = 1$.

Our function now becomes

$$y = 5.45 \cos\left(\frac{\pi}{6}(x-1)\right) + 12.05.$$

Finally, we plot our function and the data together.



□

Remark 2.3.10 Steps for Deriving Sinusoidal Models from Data.

1. Graph the data points.

2. Determine the characteristics of the data, including vertical and horizontal stretching, phase shifts, vertical shifts, and reflections about the x-axis or the y-axis.

3. *Amplitude*: Calculate the amplitude using the formula

$$|A| = \frac{\text{maximum} - \text{minimum}}{2}.$$

   Use the positive value if there is no reflection about the x-axis, and use the negative value if there is a reflection.

4. *Vertical Shift (D)*: The vertical shift is the average of the maximum and minimum values of the data, which corresponds to the midline of the graph. We can calculate the vertical shift using the formula:

$$D = \frac{\text{maximum} + \text{minimum}}{2}.$$

5. *Period*: Calculate the period from the data, then find

$$|B| = \frac{2\pi}{\text{period}}.$$

   Use the positive value if there is no reflection about the y-axis, and use the negative value if there is a reflection.

6. *Phase Shift*: Determine the phase shift, C, by calculating the distance between points from the data and their corresponding points on the sine or cosine function, such as the maximum, minimum, or values on the midline.

2.3.4 Finding Sinusoidal Equations with Technology

Some graphing utilities, such as the TI-83 calculator or Desmos Graphing Calculator[1], have functions that allow us to find a sinusoidal best-fit function given data. While some devices can find the best-fit for both sine and cosine functions, others calculate the best-fit line only for sine. For example, the TI-83 uses the `SinReg` function to calculate the sine function.

Example 2.3.11 Finding the Sinusoidal Equation using Desmos. Utilize a graphing utility to determine the best-fit cosine function for the data provided in Table 2.3.9.

Solution.

1. Open a new table in the Desmos Graphing Calculator by either typing "table" in a blank expression line or by clicking the Add Item menu in the upper left corner and selecting Table.

2. Enter the values from Table 2.3.9 into the table, where x_1 represents the month and y_1 represents the temperature.

3. Use the Zoom Fit icon (a magnifying glass with a + symbol) at the bottom left corner of the table to automatically adjust the graph settings window to best display the data.

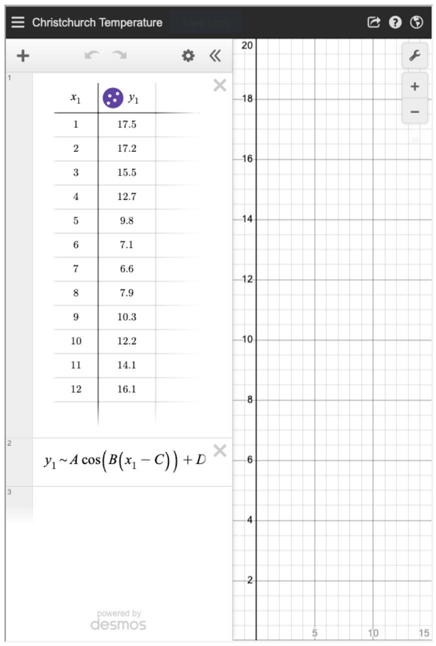
4. In a blank expression line, type "$y_1 \sim A\cos(B(x_1 - C)) + D$" to fit a cosine function to the data.

[1] `DesmosGraphingCalculator`

5. The parameters for the best-fit function will be returned: $A = 5.21412$, $B = 0.534145$, $C = 1.19926$, $D = 12.1509$. Thus, the best-fit cosine function is:

$$y = 5.21412 \cos(0.534145(x - 1.19926)) + 12.1509.$$

Figure 2.3.12 displays an interactive Desmos Graphing Calculator with the completed table and the curve of best fit plotted together.



Standalone

Figure 2.3.12 Given data points in a table, Desmos can create a sinusoidal function to model the data. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

□

2.3.5 Exercises

Exercise Group. Write the equation of a sine function with the following characteristics:

1. Amplitude: 2; Period: π.

Answer. $y = 2\sin(2x)$

2. Amplitude: 3; Period: $\frac{\pi}{2}$.

Answer. $y = 3\sin(4x)$

3. Amplitude: 1.5; Period: $\frac{\pi}{6}$; Reflection about the x-axis.

Answer. $y = -1.5\sin(12x)$

4. Amplitude: 2; Period: 3π; Reflection about the x-axis.

Answer. $y = -2\sin\left(\frac{2}{3}x\right)$

5. Amplitude: 4; Period: $\frac{2\pi}{3}$; Vertical Shift 2.

Answer. $y = 4\sin(3x) + 2$

6. Amplitude: 3; Period: $\frac{5}{2}$; Vertical Shift: $-\frac{2}{3}$.

Answer. $y = 3\sin\left(\frac{4\pi}{5}x\right) - \frac{2}{3}$

7. Amplitude: $\frac{1}{2}$; Period: 2π; Phase Shift: $\frac{\pi}{4}$.

Answer. $y = \frac{1}{2}\sin\left(x - \frac{\pi}{4}\right)$

8. Amplitude: 1; Period: $\frac{5\pi}{3}$; Phase Shift: $\frac{\pi}{6}$.

Answer. $y = \sin\left(\frac{6}{5}\left(x - \frac{\pi}{6}\right)\right)$

9. Amplitude: 2, Period: $\frac{3\pi}{2}$; Phase Shift: $\frac{\pi}{3}$; Vertical Shift: -1.

Answer. $y = 2\sin\left(\frac{4}{3}\left(x - \frac{\pi}{3}\right)\right) - 1$

10. Amplitude: 2, Period: 20; Phase Shift: $-\frac{\pi}{4}$; Vertical Shift: 3.

Answer. $y = 2\sin\left(\frac{\pi}{10}\left(x + \frac{\pi}{4}\right)\right) + 3$

Exercise Group. Write the equation of a cosine function with the following characteristics:

11. Amplitude: 1.8; Period: 2π.

Answer. $y = 1.8\cos(x)$

12. Amplitude: 2.5; Period: $\frac{\pi}{4}$.

Answer. $y = 2.5\cos(8x)$

13. Amplitude: 2; Period: $\frac{7\pi}{4}$; Reflection in the x-axis.

Answer. $y = -2\cos\left(\frac{8}{7}x\right)$

14. Amplitude: 1.5; Period: $\frac{4\pi}{5}$; Reflection in the x-axis.

Answer. $y = -1.5\cos\left(\frac{5}{2}x\right)$

15. Amplitude: 4; Period: $\frac{5\pi}{3}$; Vertical Shift: -3.

Answer. $y = 4\cos\left(\frac{6}{5}x\right) - 3$

16. Amplitude: 3.5; Period: $\frac{4\pi}{3}$; Vertical Shift: 2.

Answer. $y = 3.5\cos\left(\frac{3}{2}x\right) + 2$

17. Amplitude: 3; Period: $\frac{3\pi}{2}$; Phase Shift: $\frac{\pi}{6}$.

Answer. $y = 3\cos\left(\frac{4}{3}\left(x - \frac{\pi}{6}\right)\right)$

18. Amplitude: 2.5; Period: $\frac{\pi}{3}$; Phase Shift: $\frac{\pi}{4}$.

Answer. $y = 2.5\cos\left(6\left(x - \frac{\pi}{4}\right)\right)$

19. Amplitude: 2; Period: 3π; Phase Shift: $-\frac{\pi}{2}$; Vertical Shift: 2.

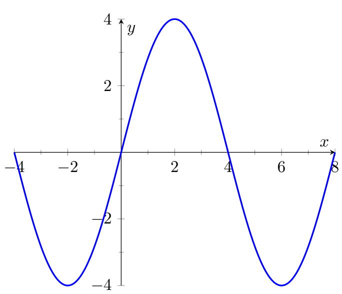
Answer. $y = 2\cos\left(\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right) + 2$

20. Amplitude: 1.2; Period: $\frac{5\pi}{2}$; Phase Shift: $-\frac{\pi}{3}$; Vertical Shift: 3.

Answer. $y = 1.2\cos\left(\frac{4}{5}\left(x + \frac{\pi}{3}\right)\right) + 3$

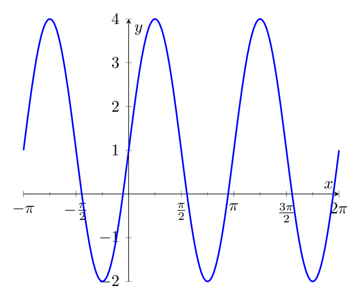
Exercise Group. For each given graph, identify the amplitude, period, phase shift, and vertical shift, and write an equation of the form $y = A\sin(B(x - C)) + D$ that represents these characteristics.

21.



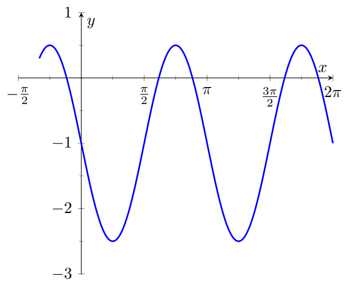
Answer. Amplitude: $A = 4$; Period: 8; Phase Shift: $C = 0$; Vertical Shift: $D = 0$; $y = 4\sin\left(\frac{\pi}{4}x\right)$

22.



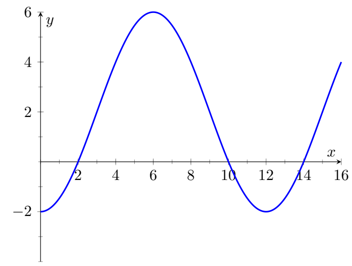
Answer. Amplitude: $A = 3$; Period: π; Phase Shift: $C = 0$; Vertical Shift: $D = 1$; $y = 3\sin(2x) + 1$

23.



Answer. Amplitude: $A = 1.5$; Period: π; Phase Shift: $C = \frac{\pi}{2}$; Vertical Shift: $D = -1$; $y = 1.5 \sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 1$

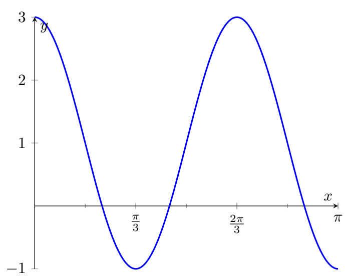
24.



Answer. Amplitude: $A = 4$; Period: 12; Phase Shift: $C = 3$; Vertical Shift: $D = 2$; $y = 4 \sin\left(\frac{\pi}{6}(x - 3)\right) + 2$

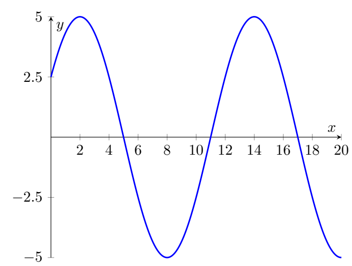
Exercise Group. For each given graph, identify the amplitude, period, phase shift, and vertical shift. Write an equation that represents these characteristics of the form $y = A\cos(B(x - C)) + D$.

25.



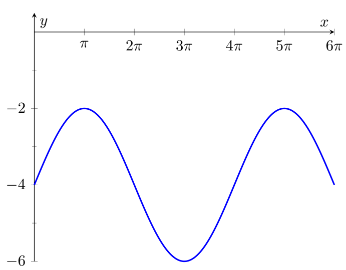
Answer. Amplitude: $A = 2$; Period: $\frac{2\pi}{3}$; Phase Shift: $C = 0$; Vertical Shift: $D = 1$; $y = 2\cos(3x) + 1$

26.



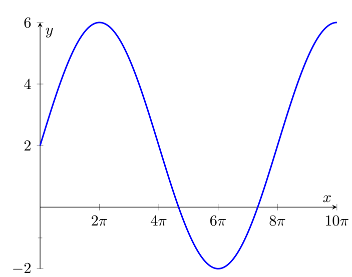
Answer. Amplitude: $A = 5$; Period: 12; Phase Shift: $C = 2$; Vertical Shift: $D = 0$; $y = 5\cos\left(\frac{\pi}{6}(x - 2)\right)$

27.



Answer. Amplitude: $A = 2$; Period: 4π; Phase Shift: $C = \pi$; Vertical Shift: $D = -4$; $y = 2\cos\left(\frac{1}{2}(x - \pi)\right) - 4$

28.

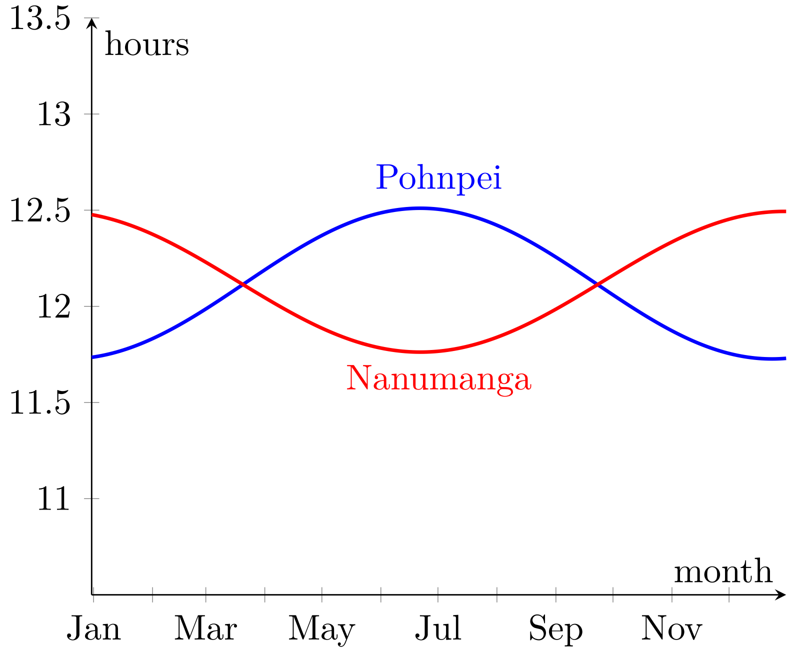


Answer. Amplitude: $A = 4$; Period: 8π; Phase Shift: $C = 2\pi$; Vertical Shift: $D = 2$; $y = 4\cos\left(\frac{1}{4}(x - 2\pi)\right) + 2$

Hours of daylight. For each of the following questions, the number of daylight hours for a pair of islands in 2025 is given. These islands share the same latitude or are close to it, with one island located north of the equator and the other south of it. Find a sinusoidal function of the form $y = A\cos(B(x - C)) + D$ to model the daylight hours for each island. This data is sourced from the NOAA Solar Calculator.

29. Pohnpei, situated at 6.9° North latitude in the Federated States of Micronesia, experiences its longest day, lasting 12.52 hours, on 21 June 2025 (the 172nd day of the year), and its shortest day, lasting 11.72 hours, on 22 December 2025 (the 356th day of the year).

Nanumanga, located at 6.3° South latitude in Tuvalu, experiences its longest day, lasting 12.49 hours, on 22 December 2025 (the 356th day of the year), and its shortest day, lasting 11.76 hours, on 21 June 2025 (the 172nd day of the year).

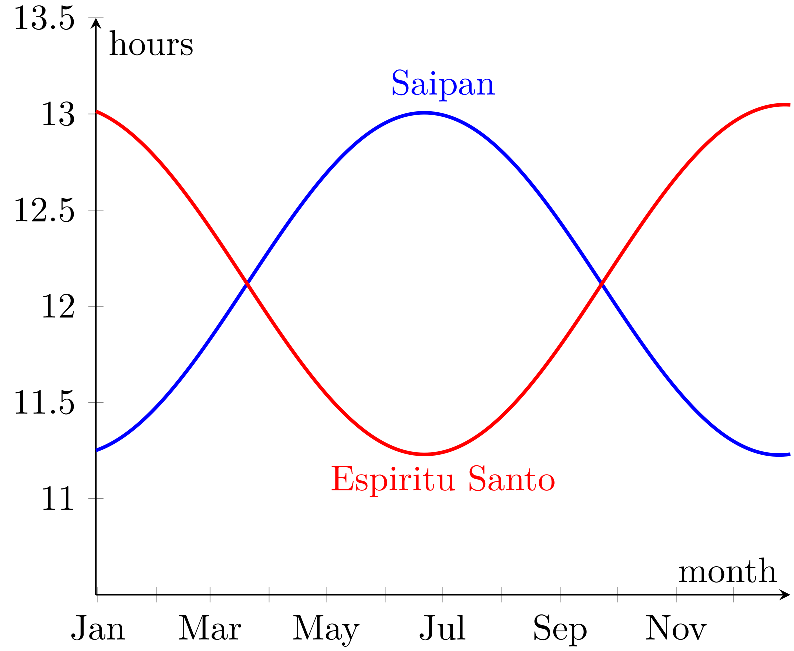


Answer. Pohnpei: $y = 0.4\cos(0.0172(x-172)) + 12.12$; Nanumanga: $y = 0.365\cos(0.0172(x-356)) + 12.125$

30. Saipan, positioned at 15.2° North latitude in the Northern Mariana Islands, has its longest day, lasting 13.03 hours, on 21 June 2025 (the 172nd day of the year), and its shortest day, lasting 11.23 hours, on 22 December 2025 (the 356th day of the year).

Espiritu Santo, located at 15.4° South latitude in Vanuatu, experiences its longest day, lasting 13.04 hours, on 22 December 2025 (the 356th day of the year), and its shortest day, lasting 11.21 hours, on 21

June 2025 (the 172nd day of the year).



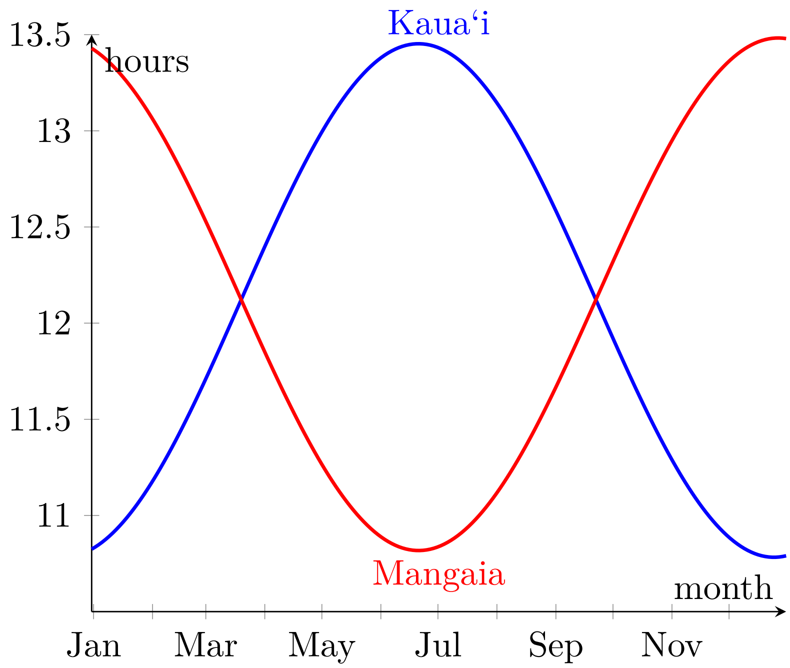
Answer. Saipan: $y = 0.9\cos(0.0172(x-172))+12.13$; Espiritu Santo: $y = 0.915\cos(0.0172(x-356)) + 12.125$

31. Kaua'i, situated at 22.1° North latitude in Hawai'i, has its longest day, lasting 13.48 hours, on 20 June 2025 (the 171st day of the year), and its shortest day, lasting 10.78 hours, on 21 December 2025 (the 355th day of the year).

Mangaia, located at 21.9° South latitude in the Cook Islands, experiences its longest day, lasting 13.47 hours, on 21 December 2025 (the 355th day of the year), and its shortest day, lasting 10.79 hours, on 20 June 2025 (the 171st day of the year).

It's noteworthy that Kaua'i and Mangaia are situated east of the International Date Line, causing them to experience the winter and summer solstice one day earlier than islands located west of the

International Date Line.



Answer. Kaua'i: $y = 1.35\cos(0.0172(x - 171)) + 12.13$; Mangaia: $y = 1.34\cos(0.0172(x - 355)) + 12.13$

32. What patterns do you observe when comparing the graphs of daylight hours for pairs of islands across different latitudes? Additionally, how does the variation in daylight hours change with increasing latitude, that is, as one moves away from the equator?

Answer. The islands exhibit mirrored patterns in daylight hours, where one island experiences longer days while the other experiences shorter days. This is due to their opposite positions relative to the equator.

As latitude increases (moving away from the equator), the variation in daylight hours also increases. Islands closer to the equator experience less variation in daylight hours throughout the year, while islands further away from the equator experience more changes in daylight hours between seasons.

Exercise Group. The islands of O'ahu and Rarotonga are located at similar distances from the equator. However, they experience different climates due to their locations relative to the equator. O'ahu is at 21.26° North, directly mirrored by Rarotonga at 21.26° South. The table below gives the average monthly temperatures (in °C) for each island. Use the data in the table to answer the following questions. Source: http://www.worldclimate.com, retrieved on 18 March, 2024.

Month (x)	Oʻahu (21.3° N)	Rarotonga (21.2° S)
January (1)	22.7	25.7
February (2)	22.7	26.1
March (3)	23.5	25.8
April (4)	24.3	24.9
May (5)	25.2	23.5
June (6)	26.3	22.3
July (7)	26.9	21.7
August (8)	27.4	21.6
September (9)	27.2	22.1
October (10)	26.4	22.9
November (11)	25.1	23.8
December (12)	23.3	24.8

33. Determine a sinusoidal function of the form $y = A\sin(B(x - C)) + D$ to represent the average monthly temperatures provided in the table for each island.

Answer. Oʻahu: $y = 2.35\sin(0.5236(x - 5)) + 25.05$; Rarotonga: $y = 2.25\sin(0.5236(x - 11)) + 23.85$

34. Utilize a graphing utility to identify the best-fit sinusoidal function of the form $y = A\sin(B(x - C)) + D$ for each island.

Answer. Oʻahu: $y = 2.3727\sin(0.502443(x - 4.72583)) + 25.005$; Rarotonga: $y = 2.24202\sin(0.521233(x + 1.16396)) + 23.7744$

35. How do the temperature patterns of Oʻahu, situated in the northern hemisphere, compare with those of Rarotonga, positioned in the southern hemisphere?

Answer. During Oʻahu's winter, Rarotonga experiences its summer, and similarly, during Oʻahu's summer, Rarotonga experiences its winter.

Exercise Group. Use the table below, which gives the average monthly temperatures (in Celsius) at various latitudes in the South Pacific, to answer the following questions. Express your answers in the form $y = A\sin(B(x - C)) + D$. Source: Data for Apia, Suva, Nukuʻalofa, and Rapa Nui obtained from `http://www.worldclimate.com`, retrieved on 18 March 2024; Data for Whangārei and Dunedin obtained from National Institute of Water and Atmospheric Research (NIWA)[2], retrieved on 18 March 2024.

[2] `NationalInstituteofWaterandAtmosphericResearch(NIWA)`

Month	Apia 13.8°S	Suva 18.1°S	Nukuʻalofa 21.1°S	Rapa Nui 27.1°S	Whangārei 35.7°S	Dunedin 45.9°S
Jan (1)	27.6	26.7	25.6	23.3	19.9	15.3
Feb (2)	27.6	26.9	26	23.7	20.2	15
Mar (3)	27.8	26.7	25.8	23.1	18.8	13.7
Apr (4)	27.8	26	24.9	21.9	16.6	11.7
May (5)	27.4	24.8	23.1	20.1	14.4	9.3
Jun (6)	27.1	24	22.4	18.9	12.4	7.3
Jul (7)	26.7	23.2	21.3	18	11.6	6.6
Aug (8)	26.5	23.2	21.2	17.9	11.9	7.7
Sep (9)	26.7	23.7	21.7	18.3	13.3	9.5
Oct (10)	27	24.4	22.4	19	14.6	10.9
Nov (11)	27.4	25.2	23.5	20.4	16.4	12.4
Dec (12)	27.4	26.1	24.7	21.8	18.5	13.9

36. Determine a sinusoidal function to represent the average monthly temperature in

(a) Apia, Sāmoa (13.8°S)

Answer. $y = 0.65\sin(0.5236(x - 11)) + 27.15$

(b) Suva, Fiji (18.1°S)

Answer. $y = 1.85\sin(0.5236(x - 11)) + 25.05$

(c) Nukuʻalofa, Tonga (21.1°S)

Answer. $y = 2.4\sin(0.5236(x - 11)) + 23.6$

(d) Rapa Nui (27.1°S)

Answer. $y = 2.9\sin(0.5236(x - 11)) + 20.8$

(e) Whangārei, New Zealand (35.7°S)

Answer. $y = 4.3\sin(0.5236(x - 11)) + 15.9$

(f) Dunedin, New Zealand (45.9°S)

Answer. $y = 4.35\sin(0.5236(x - 11)) + 10.95$

37. Utilize a graphing utility to identify the best-fit sinusoidal function in

(a) Apia, Sāmoa (13.8°S)

Answer. $y = 0.599155\sin(0.590429(x - 0.197373)) + 27.2133$

(b) Suva, Fiji (18.1°S)

Answer. $y = 1.86982\sin(0.532431(x + 1.0423)) + 25.0509$

(c) Nukuʻalofa, Tonga (21.1°S)

Answer. $y = 2.41339\sin(0.521333(x + 1.07987)) + 23.5577$

(d) Rapa Nui (27.1°S)

Answer. $y = 2.93283\sin(0.494705(x + 1.55737)) + 20.6621$

(e) Whangārei, New Zealand (35.7°S)

Answer. $y = 4.27205\sin(0.503071(x + 1.88988)) + 15.8716$

(f) Dunedin, New Zealand (45.9°S)

Answer. $y = 4.07644\sin(0.524652(x + 1.8183)) + 11.1006$

38. Based on the sinusoidal functions that you found, what can you conclude about the relationship between temperature and latitude?

Answer. As latitude increases (moving away from the equator towards the poles), the sinusoidal function's amplitude increases, signifying greater temperature variability, while its vertical shift decreases, indicating lower average temperatures at higher latitudes.

2.4 Inverse Trigonometric Functions

Swells are a crucial navigational tool, providing a consistent means of maintaining a straight course over extended periods. Unlike stars, which may be obscured during the day or on cloudy nights, or winds that can change direction frequently, swells tend to remain relatively constant. Crew members can navigate a straight course by keeping the angle at which the swell passes the canoe constant.

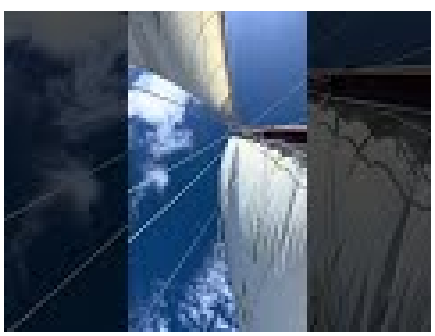
Even in poor visibility, when the crew may not see the swells, they can feel them. Parts of the vaka (canoe) lift and lower as the swells pass beneath. Skilled navigators use these movements to maintain course, even without directly seeing the waves.

The lateral (side-to-side) rocking motion of a double-hull canoe is known as **roll**. This motion occurs when a swell approaches from either the port (left) or starboard (right) side, causing the vaka to initially lift the corresponding hull (port or starboard), followed by the opposite hull, depending on the direction of the swell.

When a swell approaches from the bow (front) of the canoe, it lifts the front, causing the canoe to tilt backward before tilting forward. This motion is known as **pitch.** Conversely, if the swell comes from the stern (back), the canoe tilts forward and then backward.

When a swell approaches the canoe from an angle that is not perpendicular to any side, a combination of pitch and roll occurs. The specific motion experienced depends on the precise angle of the swell. For instance, if the swell comes from an angle and lifts the starboard bow (front right), it may subsequently lift the port bow, followed by the starboard stern, and finally the port stern. However, with a slightly different angle, the sequence may change. After lifting the starboard bow, it could lead to the starboard stern being lifted next, followed by the port bow, and finally the port stern. This twisting motion is referred to as a **corkscrew** effect due to its combination of motions.

The video in Figure 2.4.1 demonstrates how vaka Paikea moves as swells pass under the hulls. A change in vaka motion can indicate either a change in the canoe's direction or a shift in the direction of ocean swells. In such cases, the crew must assess the situation and, when conditions allow, utilize celestial markers, such as the rising and setting of stars, to determine the direction of the swells.



Standalone

Figure 2.4.1 As swells pass under vaka Paikea, the canoe pitches, rolls, and corkscrews, depending on the angle of the swell. A navigator can use these movements to keep a straight course. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to watch the video online.

As vaka Paikea sails north in the Cook Islands from Rarotonga to Aitutaki with a heading of $0°$, the swells are approaching the vaka from the southwest and moving towards the northeast. Referring to Figure 1.2.17, which provides heading angles, we observe that the swells have a heading ranging between 0° (north) and 90° (east).

Additional observations of the crew reveal that the swells hit Paikea in the following order: 1) port stern (back left); 2) starboard stern (back right); 3) port bow (front left); and 4) starboard bow (front right), as shown in Figure 2.4.2. If Paikea is 14.8 m (50 ft) in length and 6.2 m (20 ft) in width, what is the possible range of headings from which the swells may be approaching?

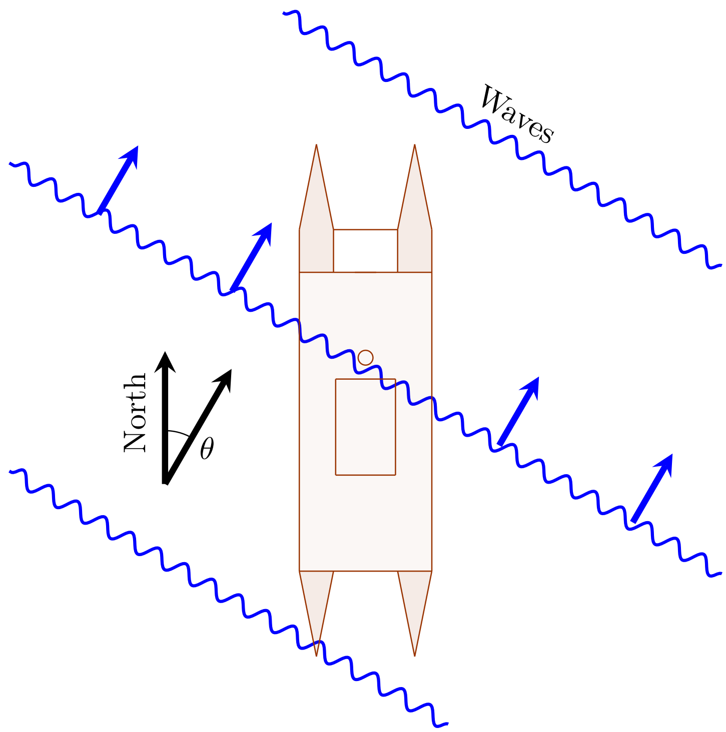


Figure 2.4.2 Swells moving towards the northeast pass under vaka Paikea heading.

To determine the range of headings, we start by considering one boundary, which occurs when the swells are moving directly north with a heading of 0°. To identify the other boundary, we want to find the angle at which the swell intersects both the starboard stern and port bow simultaneously as it passes beneath the vaka. Essentially, we want to find the angle that diagonally traverses the canoe from one corner to another. To simplify this, we represent the vaka as a rectangle and create a triangle by connecting the corners. The angle, denoted as θ, is formed between the side adjacent to the 14.8 m length of the canoe and the diagonal. This angle corresponds to the heading of the swells, as shown in Figure 2.4.3.

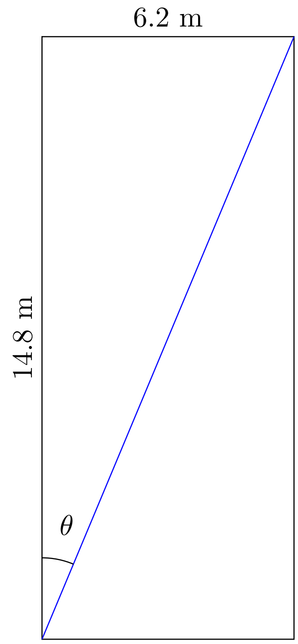


Figure 2.4.3 The deck of vaka Paikea can be simplified to a rectangle, and its diagonal represents the threshold between different sequences of motions as swells pass. The angle θ corresponds to the heading of the swells.

You may notice that for our given angle, θ, we have the opposite side of length 6.2 m and the adjacent side of length 14.8 m, which are both related to the tangent function:

$$\tan\theta = \frac{6.2}{14.8}.$$

Until now, the angles for the trigonometric problems we encountered were given. However, to solve for the angle θ, we cannot simply divide by the letters "tan", since it is part of the tangent function, which takes an angle as input and provides a ratio of sides as output. To find the angle θ, we need to use the inverse tangent function, "tan^{-1}", which takes the ratio of sides as input and gives the angle as output. To find the angle θ, we need to use the inverse function, which takes the ratio of sides as input and provides an angle as output. In this section, we will explore inverse trigonometric functions, including their properties and usage.

2.4.1 Inverse Trigonometric Functions

Recall from algebra that for a function f and its inverse function f^{-1} we have:

1. The Domain of f^{-1} = Range of f;
2. The Range of f^{-1} = Domain of f;
3. If $f(a) = b$ then $f^{-1}(b) = a$.

In terms of trigonometric functions, for example, if $f(x) = \sin x$ then $f^{-1}(x) = \sin^{-1} x$. Now consider $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, then $\frac{\pi}{4} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Remark 2.4.4 Be Careful. Do not confuse the inverse trigonometric notation with an exponent, in other words, $\sin^{-1} x \neq \frac{1}{\sin x}$. To avoid this, we will use parentheses around the trigonometric function to denote the power of negative one: $(\sin x)^{-1}$.

Also recall from algebra that for a function, f, to have an inverse, f^{-1}, it must be one-to-one, meaning no horizontal line intersects the graph more than once. Since this is not true for trigonometric functions, they do not have inverses. We need functions to be one-to-one because both $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ but if we took the inverse of sine, we wouldn't know whether to use $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ or $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$. To avoid this confusion and to ensure the function is one-to-one, we can put restrictions on the domains of each trigonometric function so they attain all the values in the range only once, making the restricted function one-to-one, and thus having an inverse (see Figure 2.4.5).

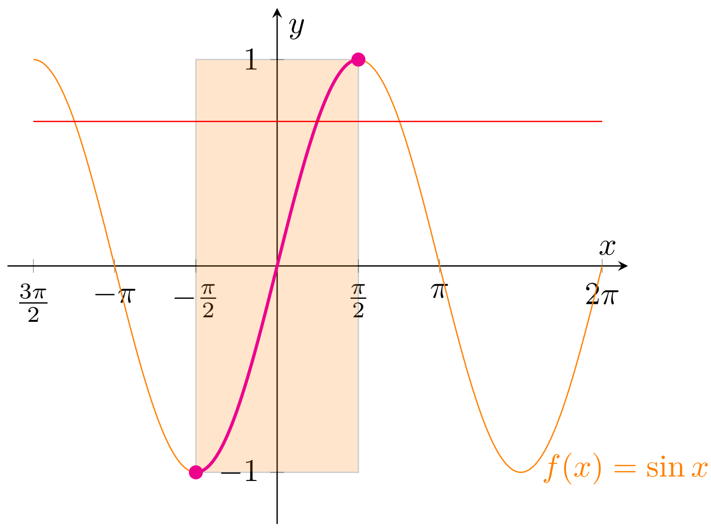


Figure 2.4.5 The graph of $f(x) = \sin x$ does not pass the horizontal line test and is not one-to-one. If we restrict the graph to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ so that each value in the range $[-1, 1]$ is attained only once, then the restricted function is one-to-one and has an inverse.

We will restrict the domain of $y = \sin x$ to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the domain of $y = \cos x$ to $[0, \pi]$, and the domain of $y = \tan x$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Notice that the domain for each trigonometric function includes one quadrant where the function is positive and one quadrant where it is negative. The domains and the corresponding graphs for sine, cosine, and tangent are shown in Figure 2.4.6, Figure 2.4.7, and Figure 2.4.8, respectively.

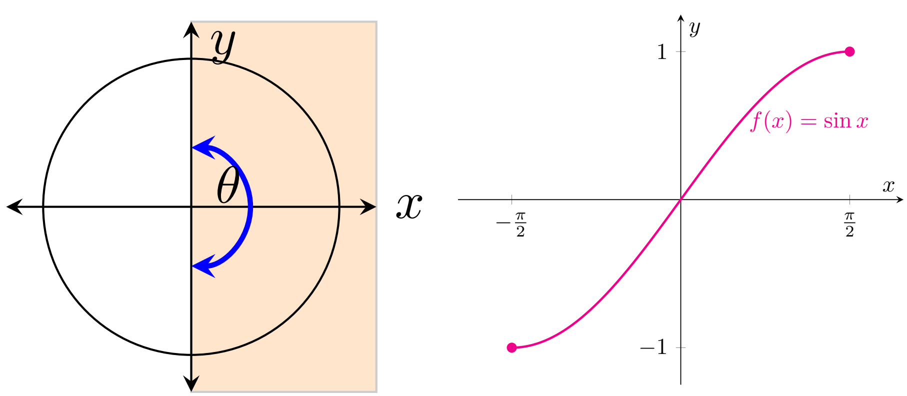


Figure 2.4.6 The domain of $y = \sin x$ (left) and its graph on the restricted domain (right).

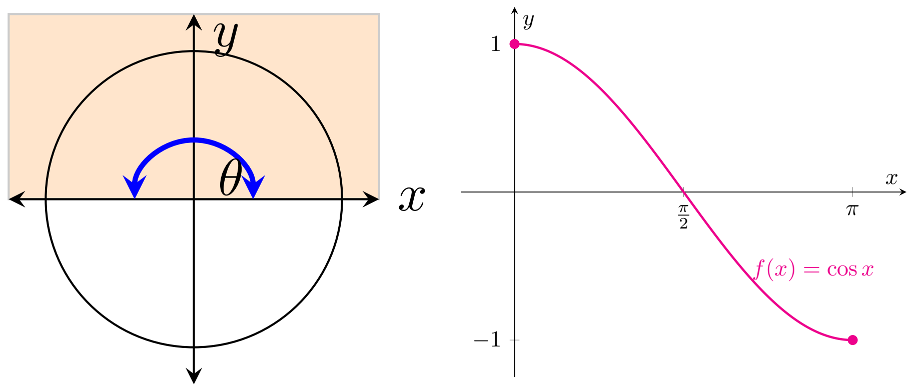


Figure 2.4.7 The domain of $y = \cos x$ (left) and its graph on the restricted domain (right).

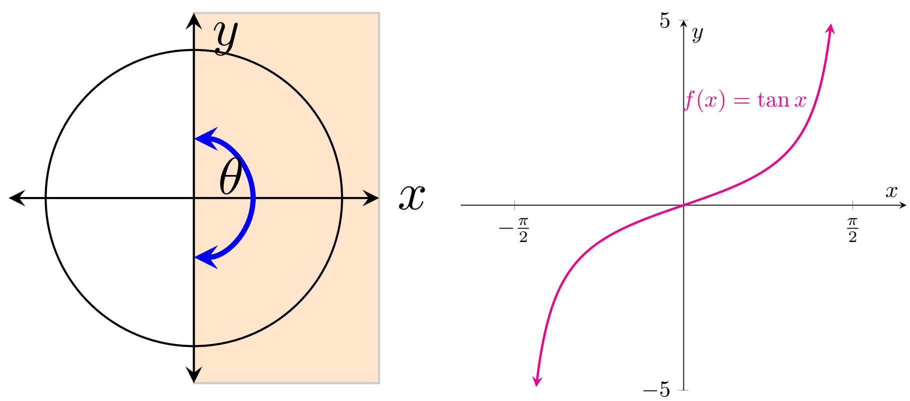


Figure 2.4.8 The domain of $y = \tan x$ (left) and its graph on the restricted domain (right).

With these restrictions on the domains, we now have trigonometric functions that are one-to-one and so we can define their inverse functions:

Definition 2.4.9 Inverse Sine. The **inverse sine function** is symbolized by

$$y = \sin^{-1} x \qquad \text{and means} \qquad x = \sin y.$$

The inverse sine function is also called the **arcsine function**, and is denoted by $\arcsin x$. ◊

Definition 2.4.10 Inverse Cosine. The **inverse cosine function** is symbolized by

$$y = \cos^{-1} x \qquad \text{and means} \qquad x = \cos y.$$

The inverse cosine function is also called the **arccosine function**, and is denoted by $\arccos x$. ◊

Definition 2.4.11 Inverse Tangent. The **inverse tangent function** is symbolized by

$$y = \tan^{-1} x \qquad \text{and means} \qquad x = \tan y.$$

The inverse tangent function is also called the **arctangent function**, and is denoted by $\arctan x$. ◊

Definition 2.4.12 Inverse Cosecant, Secant, and Cotangent. Inverse cosecant, inverse secant, and inverse cotangent functions are not as common as the other trigonometric functions and we will just summarize them.

$$y = \csc^{-1} x \quad \text{means} \quad x = \csc y,$$
$$y = \sec^{-1} x \quad \text{means} \quad x = \sec y,$$
$$y = \cot^{-1} x \quad \text{means} \quad x = \cot y.$$

◊

Definition 2.4.13 Domain and Range for Inverse Trigonometric Functions. The domain and range for each function is:

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

◊

2.4.2 Finding the Exact Value of an Inverse Trigonometric Function

Example 2.4.14 Evaluating Inverse Trigonometric Functions. Find the exact value of:

(a) $\cos^{-1}\left(\frac{1}{2}\right)$

Solution. Let $\theta = \cos^{-1}\left(\frac{1}{2}\right)$. Evaluating the problem is the same as determining the angle, θ, for which

$$\cos\theta = \frac{1}{2}.$$

Although there are infinitely many values of θ that satisfy the equation, such as $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$, there is only one value that lies in the interval $[0, \pi]$. Thus, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

(b) $\tan^{-1}(\sqrt{3})$

Solution. Let $\theta = \tan^{-1}(\sqrt{3})$. We must find θ that satisfies $\tan\theta = \sqrt{3}$ while also ensuring θ is within the range of the inverse tangent function. Because $\tan\frac{\pi}{3} = \sqrt{3}$ and $-\frac{\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2}$, we conclude that $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

(c) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution. The angle, θ, in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that satisfies $\sin\theta = -\frac{1}{\sqrt{2}}$ is $\theta = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$.

□

2.4.3 Approximations of Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not have special angles as values, we will need to use a calculator.

Remark 2.4.15 Using a Calculator. Using a calculator for inverse trigonometric functions: Most calculators will have a special key for the inverse sine, inverse cosine, and inverse tangent functions. Depending on the calculator, we may see the following keys for the inverse trigonometric functions:

Function	Calculator Key
inverse sine	[SIN^-1], [INV][SIN], [ARCSIN], or [ASIN]
inverse cosine	[COS^-1], [INV][COS], [ARCCOS], or [ACOS]
inverse tangent	[TAN^-1], [INV][TAN], [ARCTAN], or [ATAN]

Often, the inverse trigonometry key can be found by first pressing [2nd] or [SHIFT], followed by the trigonometry function key. For example, to get the [SIN^-1] key, press [2nd][SIN] or [SHIFT][SIN].

Many calculators do not have specific keys for the inverse cosecant, inverse secant, and inverse cotangent functions. Instead, we can use the Reciprocal Identities (Definition 1.4.2) to get:

Function	Calculator Key
inverse cosecant	[1][/][SIN^-1]
inverse secant	[1][/][COS^-1]
inverse cotangent	[1][/][TAN^-1]

Example 2.4.16 Finding an Approximate Value of Inverse Trigonometric Functions. Use a calculator to approximate the value of each expression in radians, rounded to two decimals.

(a) $\cos^{-1}(-0.39)$

Solution. First verify the mode of the calculator is in radians. Then, press the following keys [COS^-1][(][(-)][0.39][)][ENTER] to get

$$\cos^{-1}(-0.39) \approx 1.9714279195.$$

If the calculator does not have the [COS^-1] key, then use the appropriate key(s) for inverse cosine, such as [INV][COS], [ARCCOS], or [ACOS].

On some calculators, [COS^-1] is pressed first, then [(][(-)][0.39][)]; while other calculators the sequence is reversed with [(][(-)][0.39][)] pressed first, then [COS^-1]. Verify with the calculator's manual.

(b) $\tan^{-1}(12)$

Solution. First verify the mode of the calculator is in radians. Then, using the appropriate key for inverse tangent, press the following keys [TAN^-1][(][12][)][ENTER] to get

$$\tan^{-1}(12) \approx 1.48765509491.$$

(c) $\sin^{-1}(0.8)$

Solution. First verify the mode of the calculator is in radians. Then, using the appropriate key for inverse sine, press the following keys [SIN^-1][(][0.8][)][ENTER] to get

$$\sin^{-1}(0.8) \approx 0.927295218002.$$

□

Example 2.4.17 Swells Approaching Paikea. At the start of this section, we discussed the swells passing beneath Paikea in the following order: 1) port stern (back left); 2) starboard stern (back right); 3) port bow (front left); and 4) starboard bow (front right). Given that Paikea is 14.8 m (50 ft) in length and 6.2 m (20 ft) in width, what is the possible range of headings from which the swells may be approaching?

Solution. If the swells are moving directly north, the heading will be 0°.

To find the other boundary, we want to determine the angle at which the swell simultaneously intersects both the starboard stern and port bow as it passes beneath the vaka. This angle, denoted as θ, is formed between the side adjacent to the 14.8 m length of the canoe and the diagonal. Using the tangent function we have:

$$\tan\theta = \frac{6.2}{14.8}.$$

Solving for θ using the inverse tangent function we get:

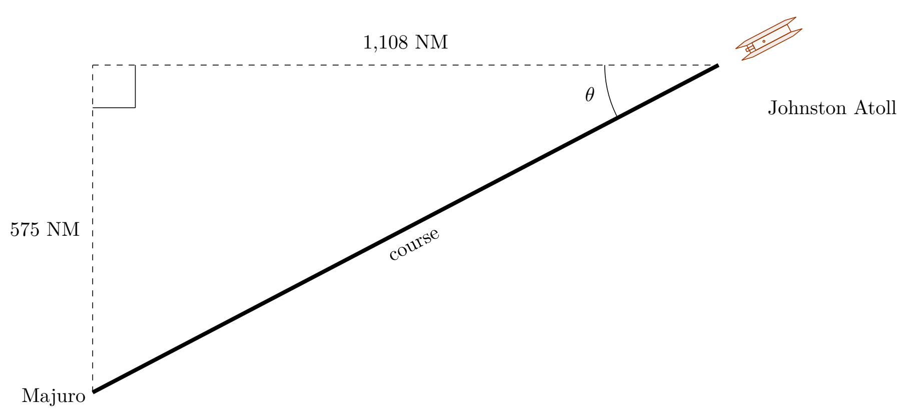
$$\tan^{-1}\left(\frac{6.2}{14.8}\right) \approx 22.73°.$$

Therefore, the possible range of headings from which the swells may be approaching is between 0° (directly north) and approximately 22.73°. □

Example 2.4.18 A gift for Mau. In 1999, after Mau Pialug, also known as Papa Mau, sailed from Hawaiʻi to his home in Satawal, Micronesia aboard the Makaliʻi, the crew of Nā Kālai Waʻa expressed their gratitude to the man who shared his knowledge of navigation by constructing a sister canoe to Makaliʻi. This vessel, named the Alingano Maisu, was a 56-foot long double-hulled voyaging canoe. In 2007, accompanied by the Hōkūleʻa, the Alingano Maisu embarked on her inaugural journey to Satawal, continuing Papa Mau's legacy of navigation in his home islands.

During this journey, the canoes sailed directly towards Johnston Atoll, using it as a sighting point without making a stop, before proceeding to their first destination in Majuro, Marshall Islands. The island of Majuro is situated 1,108 nautical miles west and 575 nautical miles south of Johnston Atoll.

What house do we need to sail in and what distance will we need to sail? If the waʻa travels at 5 knots, how many days will it take to reach the destination? Note that 1 knot = 1 nautical mile/hour.



We will use the tangent function because we are given the side opposite to θ (575 NM) and the side adjacent to θ (1,108 NM) and express it as

$$\tan\theta = \frac{575}{1,108}.$$

Solution. Since we were able to write $\tan\theta = \frac{575}{1,108}$, we can use a calculator or other technology to evaluate the inverse tangent to find our angle:

$$\theta = \tan^{-1}\left(\frac{575}{1,108}\right) \approx 27.4°.$$

Because this angle is in Quadrant III, we find its value on the Unit Circle by adding $180° + 27.4° = 207.4°$. Next we refer to the Star Compass with angles (Figure 1.2.4) to conclude we will need to sail towards the House ʻĀina Kona.

To determine the distance, d, we use the Pythagorean Theorem:

$$d = \sqrt{575^2 + 1,108^2} \approx 1,248.3\text{NM}.$$

Finally, we note that since (speed) = (distance)/(duration), we can rearrange the terms to get (duration) = (distance)/(speed). If we travel at 5 knots (5 NM/hr), we can calculate the duration as

$$\begin{aligned}
\text{duration} &= \frac{1,248.3\text{ NM}}{5\text{ knot}} && \frac{\text{distance}}{\text{speed}}\\
&= \frac{1,248.3\text{ NM}}{5\text{ NM/hr}} && 1\text{ knot}=1\text{ NM/hr}\\
&= 249.7\text{ hours} && \frac{\text{NM}}{\text{NM/hr}} = \text{NM}\cdot\frac{\text{hr}}{\text{NM}} = \text{hr}\\
&= 249.7\text{ hours}\cdot\frac{\text{day}}{24\text{ hours}}\\
&\approx 10.4\text{ days.}
\end{aligned}$$

□

Example 2.4.19 Finding Land. As Hōkūleʻa is sailing towards Rapa Nui, the navigator uses a process called **dead reckoning** to determine their position based on the latitude, measured by the stars, and other factors such as the estimated distance traveled, speed, and direction. Once the navigator has determined that they are in the vicinity of land, her attention is now focused on looking for signs of land. One method navigators will use is to look for land-based seabirds such as the manu-o-Kū (fairy tern) and the noio (noddy tern), which go out to sea in the morning to fish and return to land at night. However, Rapa Nui's seabird population has been reduced so she will look for other signs such as drifting land vegetation; clouds that form over islands; the loom of the island when white sand and still lagoons reflect the sunlight or moonlight upwards; and distinctive patterns of swells bending (refracting) around or reflecting off islands. Land will be spotted when the navigator first sees Maʻunga Terevaka, the tallest point in Rapa Nui, which stands at 1,665 ft. Figure 2.4.20 depicts the relation between the canoe and island (left) as well as what is seen from the deck of the canoe (right).

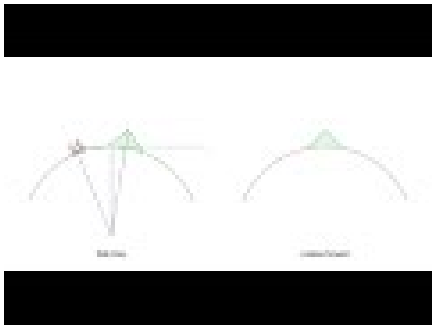


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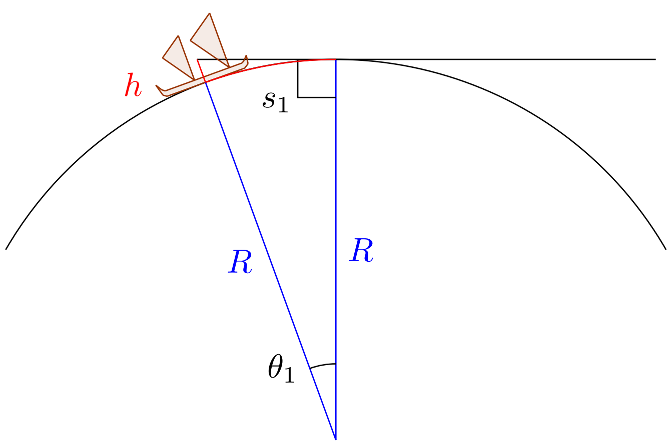
Figure 2.4.20 Watch as the canoe approaches an island from two different views. On the left, see a side view as the island gradually comes into sight over the horizon. On the right, experience the perspective from the canoe's deck, observing the island appearing to rise from the water. This dual perspective provides a unique glimpse into the legend of Māui, the demigod who pulled the islands from the ocean with his fish-hook. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to watch the video online.

1. On the deck of Hōkūleʻa, a navigator stands at 9 ft above sea level. If she looks out to the sea, how far is she from horizon? Assume the radius of the earth at Rapa Nui is 20,911,171 feet.

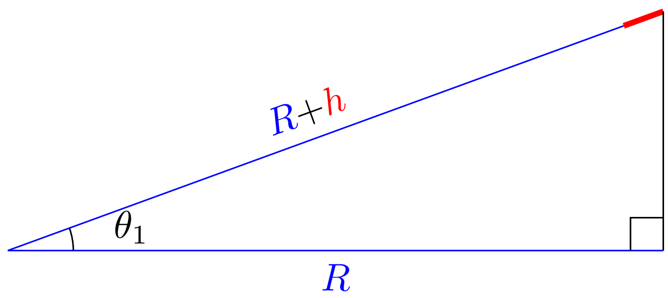
2. How far is Hōkūleʻa from Maʻunga Terevaka when it first becomes visible over the horizon to someone standing 9ft above sea level?

Solution.

1. We begin by assuming that the earth is a sphere with radius R. Standing on Hōkūleʻa, the line from the navigator's eye to the horizon is tangent to the circle of radius R.



Here $h = 9$ ft, represents the height of the navigator's eye above sea level, $R = 20,911,171$ ft is the radius of the earth at Rapa Nui, and s_1 is the arc length or distance along the surface of the earth from the navigator to the horizon. We have written the distances in feet since the height of the navigator is in feet. Recall from Theorem 1.2.26 the formula for finding the arc length is $s_1 = 2\pi R \cdot \left(\frac{\theta_1}{360}\right)$. Since we know R, we only need to find θ_1. Notice the right triangle.



Since we know the adjacent side and hypotenuse of this triangle, we can use cosine:

$$\cos\theta_1 = \frac{R}{R+h} = \frac{20,911,171}{20,911,171+9} = \frac{20,911,171}{20,911,180}.$$

To solve for θ_1, we use the inverse cosine:

$$\theta_1 = \cos^{-1}\left(\frac{20,911,171}{20,911,180}\right) \approx 0.053°.$$

Now we are ready to calculate the arc length, s_1:

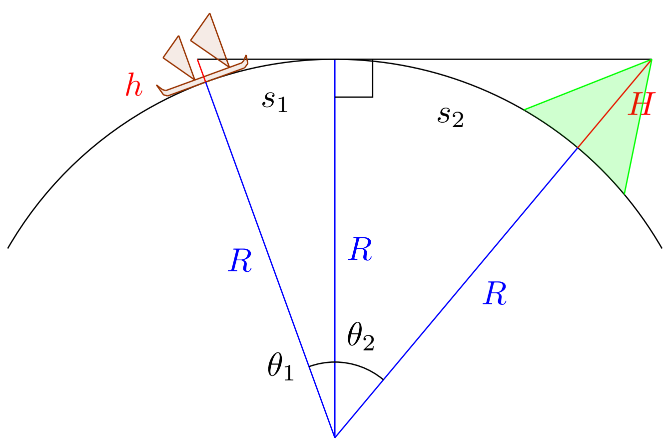
$$s_1 = 2\pi \cdot \frac{0.053°}{360°} \cdot 20,911,171 \text{ feet} \approx 19,343 \text{ feet}.$$

Converting to miles we get

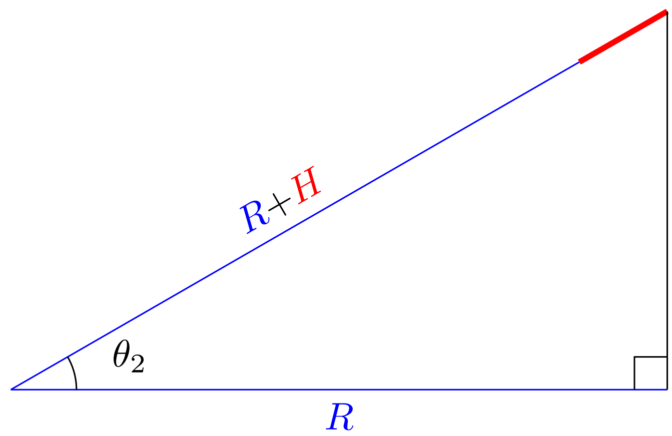
$$s_1 \approx 19,401 \text{ feet} \cdot \frac{1 \text{ mile}}{5,280 \text{ feet}} \approx 3.7 \text{ miles}.$$

So the horizon is 3.7 miles from the navigator.

2. Next, to determine how far the navigator is from Maʻunga Terevaka when it emerges over the horizon, we need to align the top of the mountain with line from the navigator's eye to the horizon.



Here, $H = 1,665$ ft is the height of Maʻunga Terevaka. To find the distance from the top of the mountain to the horizon, we will need to determine s_2. We begin by redrawing the triangle.



The angle is then given by

$$\begin{aligned}\theta_1 &= \cos^{-1}\left(\frac{R}{R+H}\right)\\ &= \cos^{-1}\left(\frac{20,911,171}{20,911,173+1,665}\right)\\ &= \cos^{-1}\left(\frac{20,911,171}{20,912,836}\right)\\ &\approx 0.723^\circ.\end{aligned}$$

We conclude that the distance from Maʻunga Terevaka to the horizon is

$$s_2 = 2\pi \cdot \frac{0.723^\circ}{360^\circ} \cdot (20,911,171 \text{ feet}) \cdot \frac{1 \text{ mile}}{5,280 \text{ feet}} \approx 50.0 \text{ miles}.$$

Therefore, the total distance between the navigator and Maʻunga Terevaka is $s_1 + s_2 = 3.7 + 50.0 = 53.7$ miles. Note that this is the distance when the island may first be seen, however, weather conditions may reduce visibility.

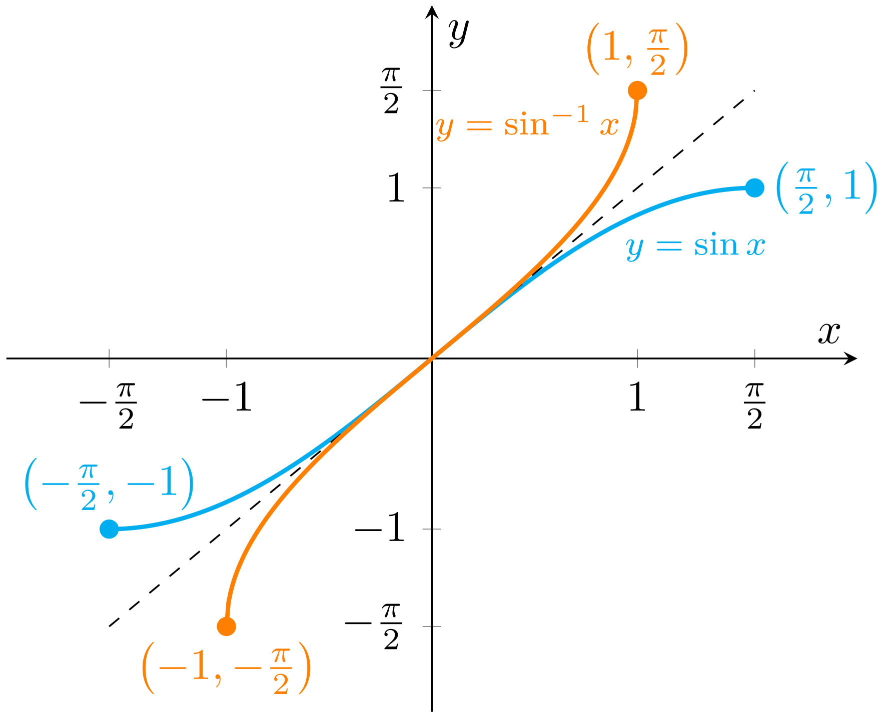
□

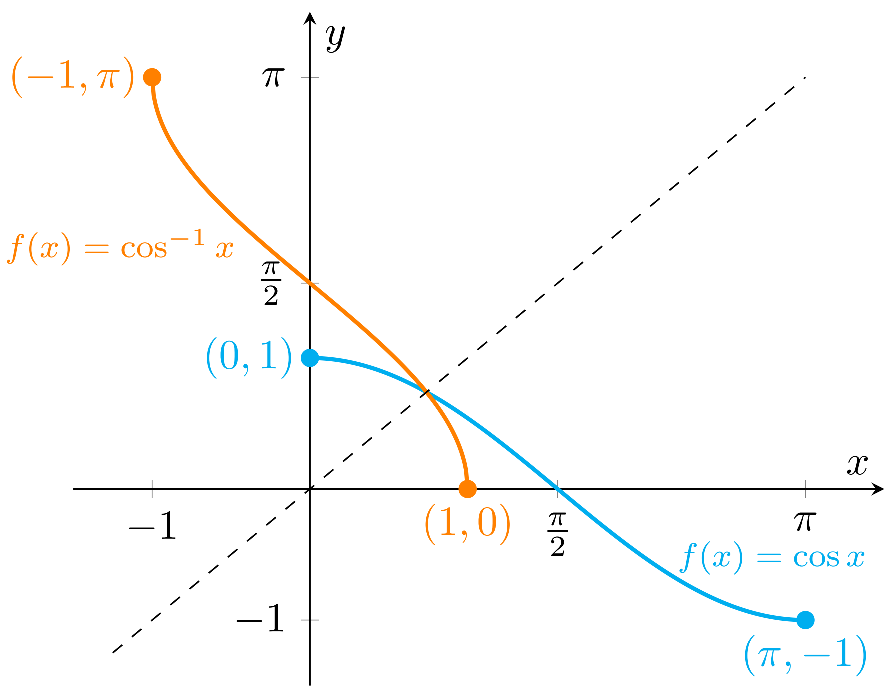
2.4.4 Graphs of Inverse Trigonometric Functions

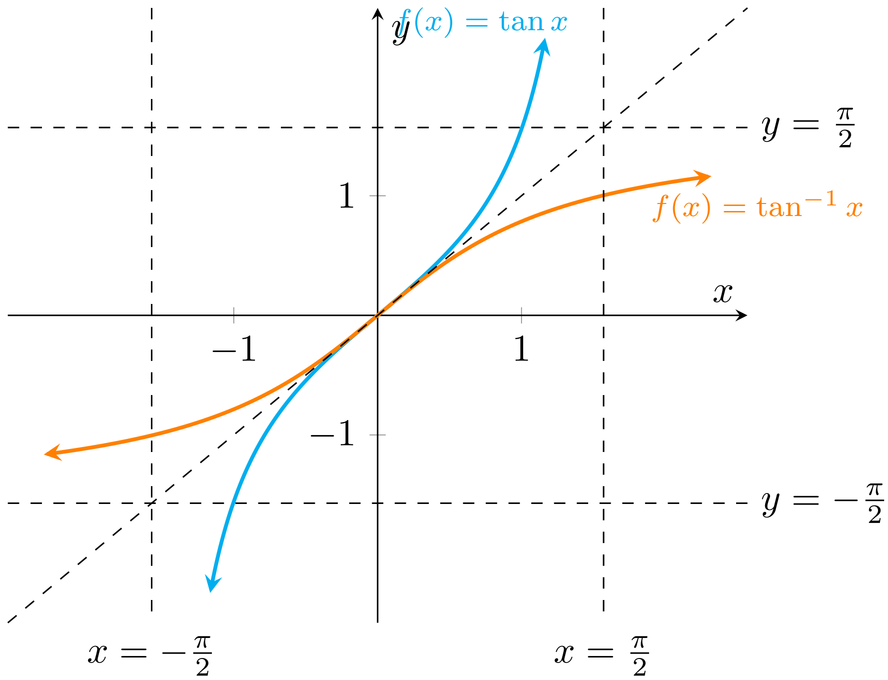
Recall from algebra that:

1. The point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.
2. The graphs of f^{-1} and f are reflections of each other about the line $y = x$.

The graph of each inverse trigonometric function can be obtained by reflecting the graph of the original function about the line $y = x$.


y
$\left(1, \frac{\pi}{2}\right)$
$\frac{\pi}{2}$
$y = \sin^{-1} x$
1
$\left(\frac{\pi}{2}, 1\right)$
$y = \sin x$
x
$-\frac{\pi}{2}$
-1
1
$\frac{\pi}{2}$
$\left(-\frac{\pi}{2}, -1\right)$
-1
$-\frac{\pi}{2}$
$\left(-1, -\frac{\pi}{2}\right)$


y
$(-1, \pi)$
π
$f(x) = \cos^{-1} x$
$\frac{\pi}{2}$
$(0, 1)$
x
-1
$(1, 0)$
$\frac{\pi}{2}$
π
$f(x) = \cos x$
-1
$(\pi, -1)$



2.4.5 Composition of Inverse Trigonometric Functions

Recall from Algebra that if f is a one-to-one function with inverse f^{-1}, then:

1. $f(f^{-1}(y)) = y$ for every y in the domain of f^{-1}.
2. $f^{-1}(f(x)) = x$ for every x in the domain of f.

In terms of trigonometric functions, $f(f^{-1}(y)) = y$ holds for all y in the domain, however, we need to be careful when evaluating $f^{-1}(f(x)) = x$ because the domain of f^{-1} is restricted.

Definition 2.4.21 Properties of Composition of Trigonometric Functions.

$\sin(\sin^{-1} x) = x$	when	$-1 \leq x \leq 1$
$\cos(\cos^{-1} x) = x$	when	$-1 \leq x \leq 1$
$\tan(\tan^{-1} x) = x$	when	$-\infty < x < \infty$
$\sin^{-1}(\sin x) = x$	*only when*	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
$\cos^{-1}(\cos x) = x$	*only when*	$0 \leq x \leq \pi$
$\tan^{-1}(\tan x) = x$	*only when*	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

◊

Example 2.4.22 Composition of a trigonometric function and the inverse of the same trigonometric function. Find the exact value of each expression:

(a) $\cos^{-1}\left(\cos \frac{\pi}{8}\right)$

Solution. Since $\frac{\pi}{8}$ is in the interval $[0, \pi]$, then from the properties of compositions of inverse functions, we get

$$\cos^{-1}\left(\cos \frac{\pi}{8}\right) = \frac{\pi}{8}.$$

(b) $\tan(\tan^{-1}(7))$

Solution. Since 7 is in the interval $(-\infty, \infty)$,

$$\tan(\tan^{-1}(7)) = 7.$$

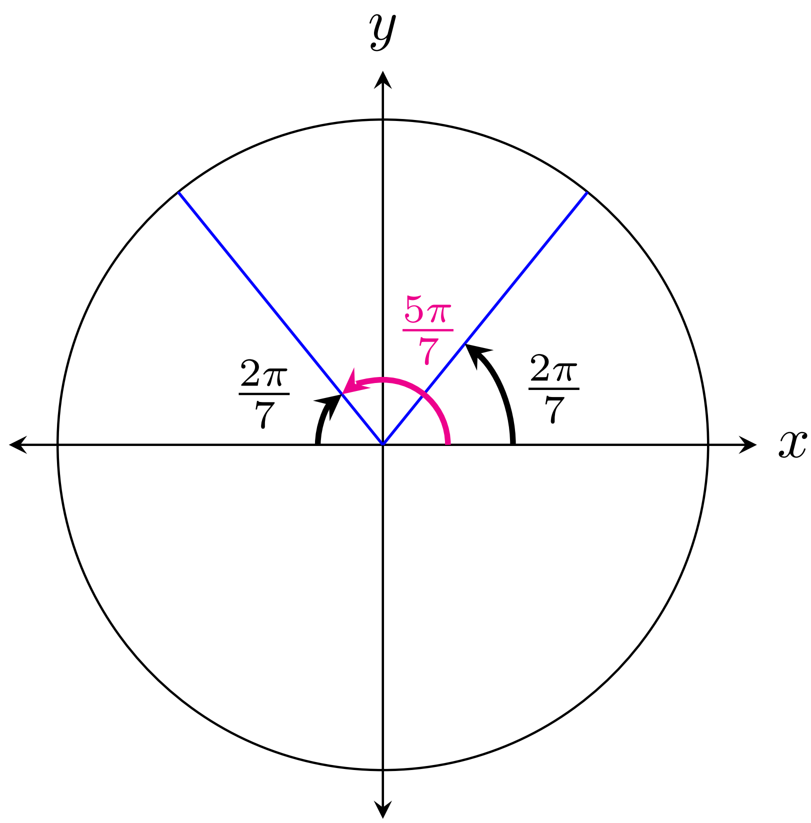
(c) $\sin^{-1}\left(\sin\left(-\frac{\pi}{7}\right)\right)$

Solution. Since $-\frac{\pi}{7}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\sin^{-1}\left(\sin\left(-\frac{\pi}{7}\right)\right) = -\frac{\pi}{7}.$$

(d) $\sin^{-1}\left(\sin\frac{5\pi}{7}\right)$

Solution. Note that $\frac{5\pi}{7}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In order to evaluate the expression, we first need to find an angle, θ, such that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\sin\frac{5\pi}{7} = \sin\theta$. Since $\frac{\pi}{2} < \frac{5\pi}{7} < \pi$, $\frac{5\pi}{7}$ is in Quadrant II. Recall from Section 1.3 that our reference angle is $\theta = \pi - \frac{5\pi}{7} = \frac{2\pi}{7}$.



Since $\frac{2\pi}{7}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\sin^{-1}\left(\sin\frac{5\pi}{7}\right) = \sin^{-1}\left(\sin\frac{2\pi}{7}\right) = \frac{2\pi}{7}.$$

(e) $\cos(\cos^{-1}(-0.283))$

Solution. Since -0.283 is in the interval $[-1, 1]$,

$$\cos(\cos^{-1}(-0.283)) = -0.283.$$

□

Now we will look at what happens when we need to evaluate the composition of a trigonometric function and the inverse of a different trigonometric function.

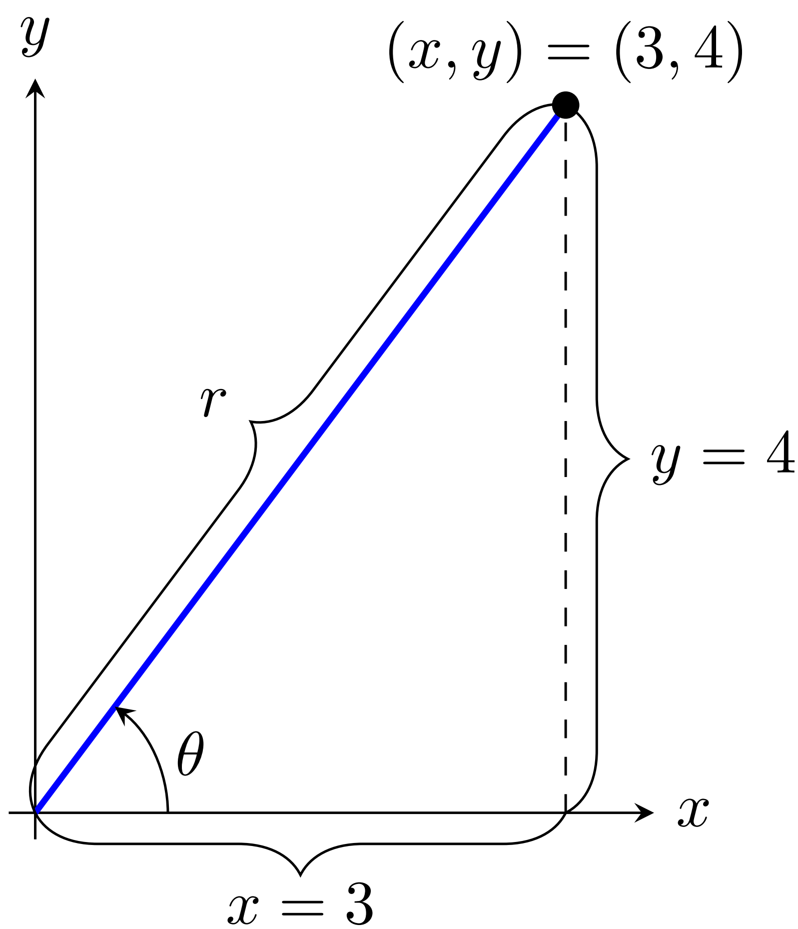
Example 2.4.23 Composition of a trigonometric function and the inverse of a different trigonometric function. Find the exact value of:

(a) $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

Solution. Let θ be an angle in the range of the inverse tangent, that is, let θ be in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta = \tan^{-1}\left(\frac{4}{3}\right)$. This equation is equivalent to $\tan\theta = \frac{4}{3}$. Since $\tan\theta > 0$, we know that θ must be in Quadrant I $\left(0 < \theta < \frac{\pi}{2}\right)$. Let (x, y) be a point on the terminal side of θ. Then by the trigonometric ratios,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

Since we have $\tan\theta = \frac{4}{3}$, we let $x = 3$ and $y = 4$, as shown in the figure below.



Evaluating $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$ is equivalent to evaluating

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{r},$$

where $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. So,

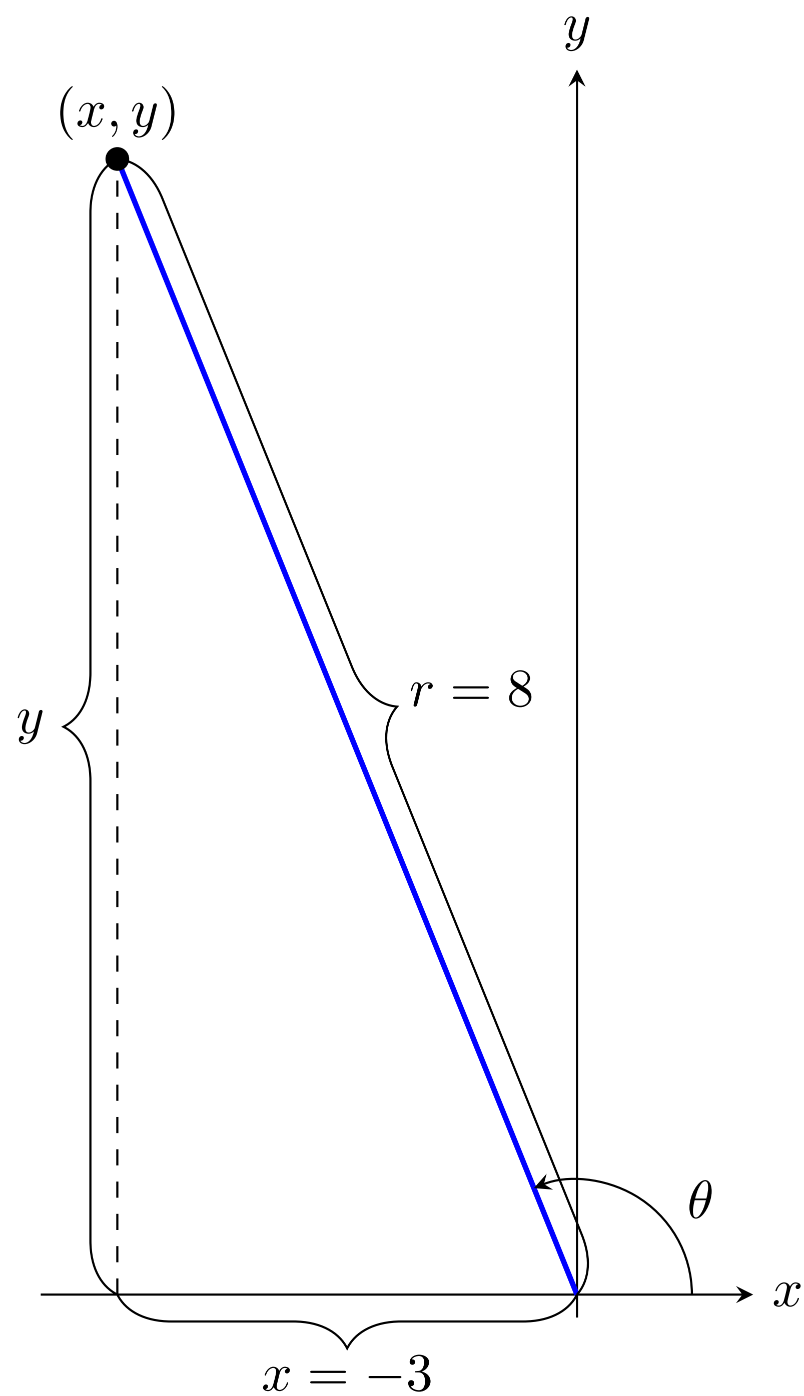
$$\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = \cos\theta = \frac{3}{5}.$$

(b) $\sin\left(\cos^{-1}\left(-\frac{3}{8}\right)\right)$

Solution. Let θ be an angle in the range of the inverse cosine, that is, let θ be in the interval $[0, \pi]$ such that $\theta = \cos^{-1}\left(-\frac{3}{8}\right)$. This equation is equivalent to $\cos\theta = -\frac{3}{8}$. Since $\cos\theta < 0$, we know that θ must be in Quadrant II $\left(\frac{\pi}{2} < \theta < \pi\right)$. Let (x, y) be a point on the terminal side of θ. Then by the trigonometric ratios,

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}.$$

Since we have $\cos\theta = -\frac{3}{8}$, we get $\frac{x}{r} = -\frac{3}{8}$ which gives us either $x = -3$ and $r = 8$ or $x = 3$ and $r = -8$. Since θ is in Quadrant II, we know that x is negative, thus $x = -3$ and $r = 8$, as shown in the figure below.



Evaluating $\sin\left(\cos^{-1}\left(-\frac{3}{8}\right)\right)$ is equivalent to evaluating

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{8},$$

where

$$\begin{aligned} r^2 &= x^2 + y^2 \\ 8^2 &= (-3)^2 + y^2 \\ 64 &= 9 + y^2 \end{aligned}$$

$$y^2 = 55.$$

Thus $y = \sqrt{55}$. So,

$$\sin\left(\cos^{-1}\left(\frac{-3}{8}\right)\right) = \sin\theta = \frac{\sqrt{55}}{8}.$$

Note that $\sin\theta$ is positive, since sine is positive in Quadrant II.

□

2.4.6 Exercises

Exercise Group. What are the domain and range of

1. $y = \sin x$

Answer. D: $-\infty < x < \infty$; R: $-1 \le y \le 1$

2. $y = \sin^{-1} x$

Answer. D: $-1 \le y \le 1$; R: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

3. $y = \cos x$

Answer. D: $-\infty < x < \infty$; R: $-1 \le y \le 1$

4. $y = \cos^{-1} x$

Answer. D: $-1 \le y \le 1$; R: $-0 \le y \le \pi$

Exercise Group. Determine the exact value of each expression in radians.

5. $\tan^{-1}(0)$

Answer. 0

6. $\tan^{-1}(-\sqrt{3})$

Answer. $-\frac{\pi}{3}$

7. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Answer. $\frac{3\pi}{4}$

8. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Answer. $\frac{\pi}{4}$

9. $\tan^{-1}(-1)$

Answer. $-\frac{\pi}{4}$

10. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer. $\frac{\pi}{3}$

11. $\cos^{-1}(-1)$

Answer. π

12. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Answer. $\frac{\pi}{4}$

13. $\sin^{-1}(1)$

Answer. $\frac{\pi}{2}$

14. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Answer. $-\frac{\pi}{3}$

15. $\cos^{-1}\left(-\frac{1}{2}\right)$

Answer. $\frac{2\pi}{3}$

16. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Answer. $\frac{\pi}{6}$

Exercise Group. Use a calculator to approximate each expression. Provide your answer in radians, rounding to two decimal places.

17. $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)$

Answer. 0.73

18. $\sin^{-1}(0.63)$

Answer. 0.68

19. $\tan^{-1}(\sqrt{7})$

Answer. 1.21

20. $\sin^{-1}\left(\frac{\sqrt{2}}{9}\right)$

Answer. 0.16

21. $\tan^{-1}(0.7)$

Answer. 0.61

22. $\cos^{-1}(-0.9)$

Answer. 2.69

23. $\sin^{-1}(-0.6)$

Answer. -0.64

24. $\cos^{-1}\left(\frac{1}{4}\right)$

Answer. 1.32

25. $\cos^{-1}(-0.4)$

Answer. 1.98

26. $\tan^{-1}(-45.6)$

Answer. -1.55

27. $\sin^{-1}(-0.8)$

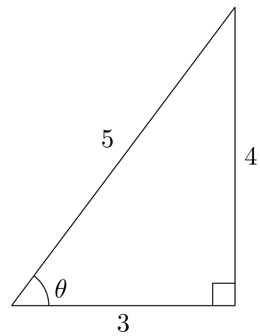
Answer. -0.93

28. $\tan^{-1}(78.9)$

Answer. 1.56

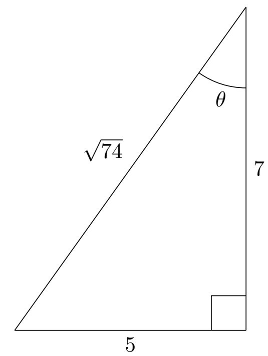
Exercise Group. In Exercise Group 1.4.7.1–8, we determined the values of the six trigonometric functions for each triangle. Now, use a calculator to find the value of θ in degrees, rounding to two decimal places.

29.



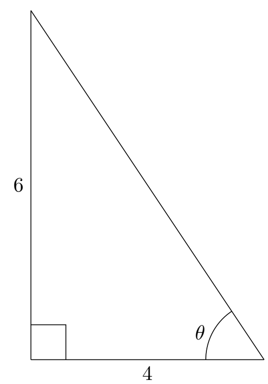
Answer. 53.13°

30.



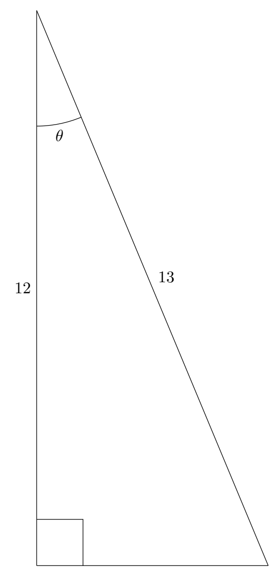
Answer. 35.54°

31.



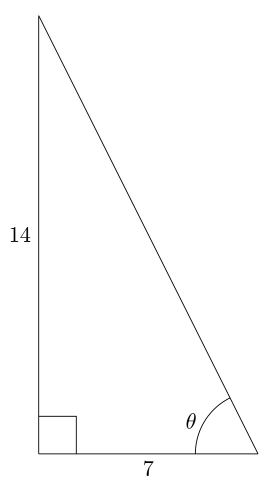
Answer. 56.31°

32.



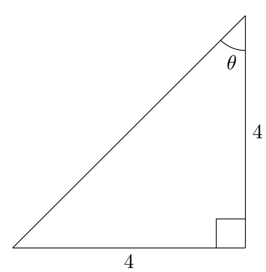
Answer. 22.62

33.



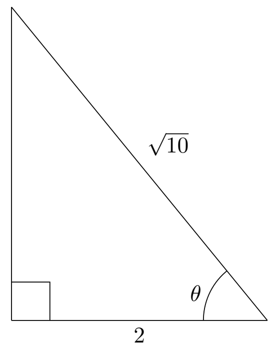
Answer. 63.43°

34.



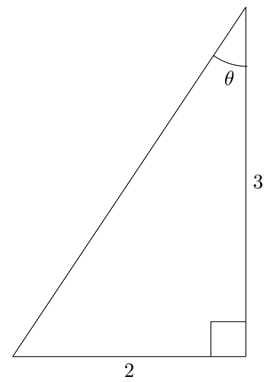
Answer. 45°

35.



Answer. $50.77°$

36.



Answer. $33.69°$

Exercise Group. Use properties of composition of trigonometric functions to find the exact value of each expression. Write "not defined" if no value exists.

37. $\sin^{-1}\left(\sin\frac{\pi}{7}\right)$
Answer. $\frac{\pi}{7}$

38. $\cos^{-1}\left(\cos\left(-\frac{\pi}{7}\right)\right)$
Answer. $\frac{\pi}{7}$

39. $\sin(\sin^{-1}(-0.8))$
Answer. -0.8

40. $\tan^{-1}\left(\tan\left(-\frac{2\pi}{9}\right)\right)$
Answer. $-\frac{2\pi}{9}$

41. $\cos(\cos^{-1}1.4)$
Answer. Not defined

42. $\sin(\sin^{-1}0.74)$
Answer. 0.74

43. $\tan(\tan^{-1}24)$
Answer. 24

44. $\cos^{-1}\left(\cos\left(-\frac{\pi}{5}\right)\right)$
Answer. $\frac{\pi}{5}$

45. $\tan(\tan^{-1}(-4.3))$
Answer. -4.3

46. $\tan^{-1}\left(\tan\frac{\pi}{8}\right)$
Answer. $\frac{\pi}{8}$

47. $\cos(\cos^{-1}0.39)$
Answer. 0.39

48. $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$
Answer. $\frac{\pi}{5}$

Exercise Group. Find the exact value of each composite function. Write "not defined" if no value exists. You may find Table 1.5.18 useful.

49. $\sin\left(\cos^{-1}\frac{1}{2}\right)$
Answer. $\frac{\sqrt{3}}{2}$

50. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$
Answer. $-\frac{1}{\sqrt{3}}$

51. $\csc\left(\tan^{-1}1\right)$
Answer. $\sqrt{2}$

52. $\sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$
Answer. $\frac{2}{\sqrt{3}}$

53. $\cos\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$
Answer. $\frac{1}{2}$

54. $\tan\left(\cos^{-1}0\right)$
Answer. Not defined

Exercise Group. Find the exact value of each composite function. Write "not defined" if no value exists.

55. $\sin\left(\tan^{-1}\frac{3}{4}\right)$
Answer. $\frac{3}{5}$

56. $\csc\left(\cos^{-1}\frac{\sqrt{3}}{7}\right)$
Answer. $\frac{7}{\sqrt{46}}$

57. $\sec\left(\tan^{-1}\left(-\sqrt{5}\right)\right)$
Answer. $\sqrt{6}$

58. $\cos\left(\sin^{-1}\frac{\sqrt{7}}{3}\right)$
Answer. $\frac{\sqrt{2}}{3}$

59. $\tan\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$
Answer. $-\frac{1}{\sqrt{8}}$

60. $\cot\left(\cos^{-1}\left(-\frac{2}{5}\right)\right)$
Answer. $-\frac{2}{\sqrt{21}}$

61. $\tan\left(\csc^{-1}\left(\frac{9}{8}\right)\right)$

Answer. $\frac{8}{\sqrt{17}}$

62. $\sec\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$

Answer. $\frac{5}{\sqrt{21}}$

Exercise Group. Utilizing the fact that $x = \frac{x}{1}$ and sketching a right triangle, find the exact value of the expression in terms of x.

63. $\cos\left(\tan^{-1} x\right)$

Answer. $\frac{1}{\sqrt{x^2+1}}$

64. $\sin\left(\cos^{-1} x\right)$

Answer. $\sqrt{1-x^2}$

65. $\sec\left(\sin^{-1} x\right)$

Answer. $\frac{1}{\sqrt{1-x^2}}$

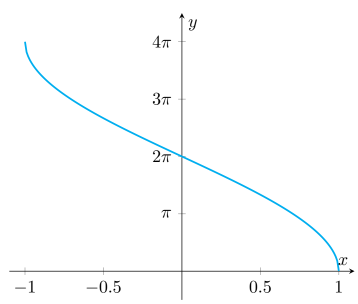
66. $\sin\left(\sec^{-1} x\right)$

Answer. $\frac{\sqrt{x^2-1}}{x}$

Exercise Group. Graph the function.

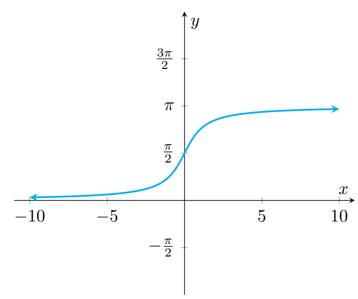
67. $y = 4\cos^{-1} x$

Answer.



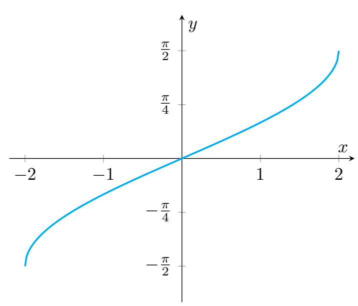
68. $y = \tan^{-1} x + \frac{\pi}{2}$

Answer.



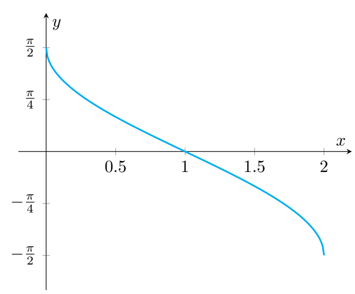
69. $y = \sin^{-1}(\frac{1}{2}x)$

Answer.



70. $y = -\sin^{-1}(x-1)$

Answer.



71. Sunrise on Mauna Kea. The summit of Mauna Kea is the highest point in Hawai‘i and sits 13,803 ft above the sea level. If we stand at the summit, we will be able to see the sun rise before someone standing at the sea level just north or south of Mauna Kea (at the same latitude). In fact, we will see the sunrise at the same time as someone at sea level sailing in a wa‘a to the east of Mauna Kea. Assume the radius of Earth is 20,917,655 feet.

(a) How far is the horizon when someone 5 ft tall stands at sea level to watch the sun rise? Round the answer to the nearest tenth of a mile.

Answer. 2.7 miles

(b) How far is the horizon when someone 5 ft tall stands on the summit to watch the sunrise? Round the answer to the nearest tenth of a mile.

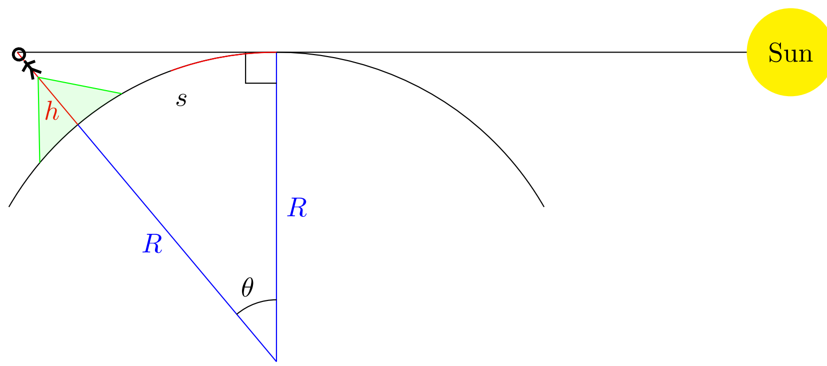
Answer. 143.9 miles

(c) How much faster would someone 5 ft tall standing on the top of the mountain see the sunrise than someone 5 ft tall standing at sea level? Express the answer in minutes, rounded to one decimal place.

Hint. The ratio of the distance to the horizon (the previous answers) to the circumference of the Earth ($2\pi R$) is equal to the ratio of the time it takes to see the sunrise to the 24 hours (1,440 minutes) of Earth's rotation. Compute the time it takes to see the sunrise from the top of the mountain and the shore, then calculate the difference.

Answer. 8.2 minutes

72. Māui Capturing the Sun. According to moʻolelo, or legend, the sun traveled very fast across the sky, leaving people with days so short there was not enough time to carry on with their daily lives. Determined to slow the sun, the demigod Māui climbed to the summit of Haleakalā, which stands at 10,023 feet, to snare the sun. Assume the radius of the Earth at Haleakalā is the same as at Mauna Kea, 20,917,655 feet.



(a) How far is the horizon when Māui stands at the summit? Assume Māui's eyes are 7 ft from the ground. Round your answer to the nearest tenth of a mile.

Answer. 122.7 miles

(b) How much sooner would Māui see the sun emerge over the horizon compared to someone whose eyes are 7 feet above sea level? Express your answer in minutes, rounded to one decimal place.

Answer. 7.1 minutes

Chapter 3

Analytic Trigonometry

3.1 Trigonometric Identities

In this chapter, we explore trigonometric identities and formulas, essential tools that enable us to algebraically manipulate and solve complex trigonometric equations. These identities and formulas enable us to analyze expressions in various forms, often simplifying complex expressions into ones that are easily solvable and interpretable. By doing so, we increase our ability to accurately model the world around us.

3.1.1 Fundamental Trigonometric Identities

An **identity** in mathematics is an equation that remains true for all valid values of its variables. We begin by reviewing some of the basic trigonometric identities from Chapter 1, collectively known as the fundamental trigonometric identities.

Definition 3.1.1 The **fundamental trigonometric identities** are:

1. **Reciprocal Identities** (Definition 1.4.2)

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

2. **Quotient Identities** (Definition 1.4.3)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

3. **Pythagorean Identities** (Definition 1.5.20)

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

4. **Even-Odd Identities** (Definition 1.5.22)

   The cosine and secant functions are **even**.

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

The sine, cosecant, tangent, and cotangent functions are **odd**.

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc(\theta)$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot(\theta)$$

5. **Cofunction Identities** (Definition 1.4.7)

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right), \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right), \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right), \qquad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

◊

3.1.2 Simplifying Trigonometric Expressions

We use a combination of trigonometric identities, formulas, and techniques from algebra to manipulate and simplify trigonometric expressions.

Example 3.1.2 Simplify
$$\tan^2(x) \cdot \csc^2(x).$$

Solution. We can simplify this expression by writing each function in terms of sine and cosine functions:

$$\tan^2(x) \cdot \csc^2(x) = \frac{\sin^2(x)}{\cos^2(x)} \cdot \frac{1}{\sin^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

□

Example 3.1.3 Simplify

$$\sin^2(x)(\cot^2(x) - 1).$$

Solution. We can simplify this expression by first using the Pythagorean Identity and then using the Reciprocal Identity:

$$\sin^2(x)(\cot^2(x) - 1) = \sin^2(x)(-\csc^2(x)) = \sin^2(x)\left(-\frac{1}{\sin^2(x)}\right) = -1.$$

□

3.1.3 Verifying Trigonometric Identities

To verify trigonometric identities, we begin with an expression on one side of the equation and manipulate it using trigonometric identities and algebraic techniques until it matches the expression on the other side.

Remark 3.1.4 Steps for Verifying Trigonometric Identities. To verify that an equation is an identity:

1. Pick an expression on one side of the equation. Often it is the more complicated expression.

2. Transform the expression using techniques such as trigonometric identities, rewriting in terms of sine and cosine functions, factoring, common denominator, or multiplying the numerator and denominator by the same

term.

3. Continue manipulating until the transformed expression matches the other side of the equation.

4. If it is difficult to make one side equal the other, try manipulating both sides separately and make them match to reach the same result.

Note: Unlike solving equations where we perform the same operation on both sides of the equal sign, when verifying trigonometric identities, we work with only one side and manipulate it to resemble the other side.

Example 3.1.5 Verify the Identity by Rewriting in Terms of Sine and Cosine. Verify the identity

$$\frac{\sin(x)}{\tan(x)} = \cos(x).$$

Solution. We use the Quotient Identity to rewrite $\tan(x)$ in terms of $\sin(x)$ and $\cos(x)$:

$$\frac{\sin(x)}{\tan(x)} = \frac{\sin(x)}{\frac{\sin(x)}{\cos(x)}} = \cancel{\sin(x)} \cdot \frac{\cos(x)}{\cancel{\sin(x)}} = \cos(x).$$

□

Example 3.1.6 Verify the Identity by Factoring. Verify the identity

$$\cos^4(x) + \sin^2(x)\cos^2(x) = \cos^2(x).$$

Solution. First notice that both terms in $\cos^4(x) + \sin^2(x)\cos^2(x)$ contain $\cos^2(x)$. Then

$$\begin{aligned}
\cos^4(x) + \sin^2(x)\cos^2(x) &= \cos^2(x) \cdot \cos^2(x) + \sin^2(x)\cos^2(x) \\
&= \cos^2(x) \cdot \big(\cos^2(x) + \sin^2(x)\big) \\
&= \cos^2(x) \cdot 1 \\
&= \cos^2(x).
\end{aligned}$$

□

Example 3.1.7 Verify the Identity by Even-Odd Properties. Verify the identity

$$\frac{\cos(x) - \sin(x)}{\cos(-x) + \sin(-x)} = 1$$

Solution. By the Even-Odd Properties, we have $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$. Thus,

$$\frac{\cos(x) - \sin(x)}{\cos(-x) + \sin(-x)} = \frac{\cos(x) - \sin(x)}{\cos(x) - \sin(x)} = 1.$$

□

Example 3.1.8 Verify the Identity by Multiplying the Numerator and Denominator by the Same Term. Verify the identity

$$\frac{\sin(x)}{\sin(x) + \cos(x)} = \frac{1}{1 + \cot(x)}.$$

Solution. Multiplying both the numerator and denominator by $\frac{1}{\sin(x)}$, we get

$$\begin{aligned}\frac{\sin(x)}{\sin(x)+\cos(x)}\cdot\frac{\frac{1}{\sin(x)}}{\frac{1}{\sin(x)}} &= \frac{\cancel{\sin(x)}\cdot\frac{1}{\cancel{\sin(x)}}}{\cancel{\sin(x)}\cdot\frac{1}{\cancel{\sin(x)}}+\cos(x)\cdot\frac{1}{\sin(x)}}\\ &= \frac{1}{1+\frac{\cos(x)}{\sin(x)}}\\ &= \frac{1}{1+\cot(x)}.\end{aligned}$$

□

Example 3.1.9 Verify the Identity by Manipulating Both Sides Separately. Verify the identity

$$\frac{1-\cos x}{1+\cos x} = (\csc x - \cot x)^2.$$

Solution. We begin by simplifying the right-hand side of the equation:

$$\begin{aligned}(\csc x-\cot x)^2 &= \csc^2 x - 2\csc x\cot x + \cot^2 x\\ &= \csc^2 x + \cot^2 x - 2\csc x\cot x\\ &= \csc^2 x + \cot^2 x - 2\frac{1}{\sin x}\frac{\cos x}{\sin x}\\ &= \csc^2 x + \cot^2 x - 2\frac{\cos x}{\sin^2 x}.\end{aligned}$$

Next, we will manipulate the left-hand side of the equation to simplify it into $\csc^2 x + \cot^2 x - 2\frac{\cos x}{\sin^2 x}$.

$$\begin{aligned}\frac{1-\cos x}{1+\cos x} &= \frac{(1-\cos x)(1-\cos x)}{(1+\cos x)(1-\cos x)}\\ &= \frac{1-2\cos x+\cos^2 x}{1-\cos^2 x}\\ &= \frac{1+\cos^2 x-2\cos x}{\sin^2 x}\\ &= \frac{1}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}-2\frac{\cos x}{\sin^2 x}\\ &= \csc^2 x+\cot^2 x-2\frac{\cos x}{\sin^2 x}.\end{aligned}$$

Thus, since the left-hand side and the right-hand side of the equation can both be manipulated to equal $\csc^2 x + \cot^2 x - 2\frac{\cos x}{\sin^2 x}$, we have established the identity. □

3.1.4 Exercises

Exercise Group. Verify the identity.

1. $\cos\theta\sec\theta = 1$

2. $\cos x\csc x = \cot x$

3. $\frac{\cos\theta\sec\theta}{\tan\theta} = \cot\theta$

4. $\frac{\cot t\tan t}{\csc t} = \sin t$

5. $(1+\tan\theta)(1-\tan\theta)+\sec^2\theta = 2$

6. $1 - \sin^2(x) = \cos^2(x)$

7. $1 - \sec^2(\theta) = -\tan^2(\theta)$

8. $\tan(t) \cdot \cot(t) = 1$

9. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

10. $(1 - \cot\theta)^2 = \csc^2\theta - 2\cot\theta$

11. $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$

12. $\sin^2 t(\csc^2 t + \sec^2 t) = \sec^2 t$

13. $\sin^2(x) - \sin^2(x)\cos^2(x) = \sin^4(x)$

14. $\sin^2(-x) + \cos^2(-x) = 1$

15. $\cos(-t) + \sin(-t) = \cos(t) - \sin(t)$

16. $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = 1$

17. $\cot^2 x(\sec^2 x - 1) = 1$

18. $(1 + \sin(t))(1 + \sin(-t)) = \cos^2 t$

19. $\tan^4\theta = \tan^2\theta\sec^2\theta - \tan^2\theta$

20. $\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = 2\sec^2 x$

21. $\frac{1}{\csc t+1} - \frac{1}{\csc t-1} = -2\tan^2 t$

22. $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\csc^2\theta$

23. $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2 + 2\cot^2\theta$

24. $\frac{1-\cos^2 x}{\cos x} = \sin x\tan x$

25. $1 - \frac{\cos^2\theta}{1+\sin\theta} = \sin\theta$

26. $\sec^2 t + \csc^2 t = \csc^2 t\sec^2 t$

27. $\frac{1+\tan x}{1-\tan x} = \frac{\cot x+1}{\cot x-1}$

28. $\frac{\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta}{\cos\theta}$

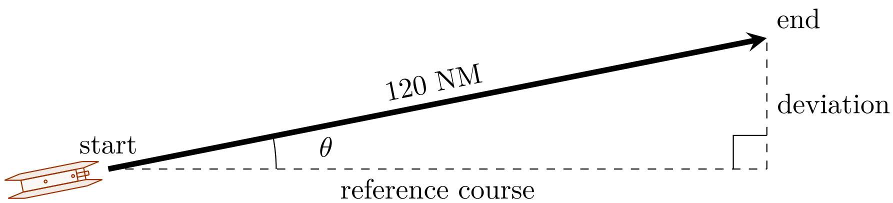
29. $\frac{\tan^2 t}{\sec t+1} = \frac{1-\cos t}{\cos t}$

30. $\frac{1+\cos\theta}{\cos\theta} = \frac{\tan^2\theta}{\sec\theta-1}$

3.2 Addition and Subtraction Formulas

In Section 1.4, we used right triangles to determine the deviation of a waʻa (canoe) from its course based on the angle of deviation. If a waʻa sails for 120 nautical miles (NM), we were able to calculate the deviation from its course using right triangles to get the equation:

$$\text{deviation} = (120 \text{ NM}) \cdot \sin(\theta).$$



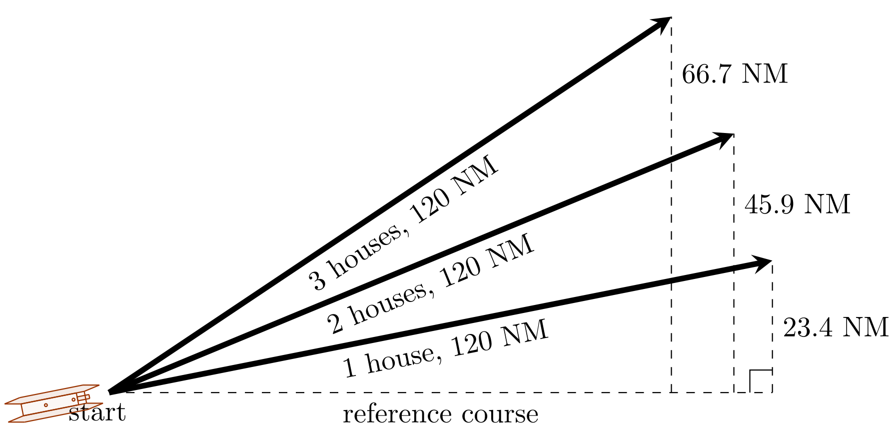
Before setting sail, a voyager studies a table listing the deviation distances corresponding to different houses of deviation. It's crucial to understand that while adding angles may yield a third angle, adding their corresponding deviations will not accurately determine the total deviation. In other words:

$$\sin(\alpha) + \sin(\beta) \neq \sin(\alpha + \beta)$$

for certain angles of deviation α and β.

To illustrate this, let's calculate the deviation distances for 1, 2, and 3 houses respectively. Using the given formula, we have:

$$\begin{aligned}
120 \sin(1 \text{ house}) &= 120 \sin(11.25^\circ) \approx 23.4 \text{ NM} \\
120 \sin(2 \text{ houses}) &= 120 \sin(22.5^\circ) \approx 45.9 \text{ NM} \\
120 \sin(3 \text{ houses}) &= 120 \sin(33.75^\circ) \approx 66.7 \text{ NM}
\end{aligned}$$



However,

$$120 \sin(1 \text{ house}) + 120 \sin(2 \text{ houses}) \approx 23.4 + 45.9 \text{ NM} \approx 69.3 \text{ NM},$$

which differs from the actual deviation of 66.7 NM when deviating by 3 houses.

These calculations demonstrate that the deviation distances for multiple houses cannot be determined by simply adding individual deviations, highlighting the importance of understanding trigonometric principles for accurate navigation. In this section, we will explore the formulas for the addition and subtraction of angles in trigonometric functions.

3.2.1 Addition and Subtraction Formulas for Cosine

First we will derive the addition and subtraction formulas for the cosine function.

Definition 3.2.1 Addition and Subtraction Formulas for Cosine.

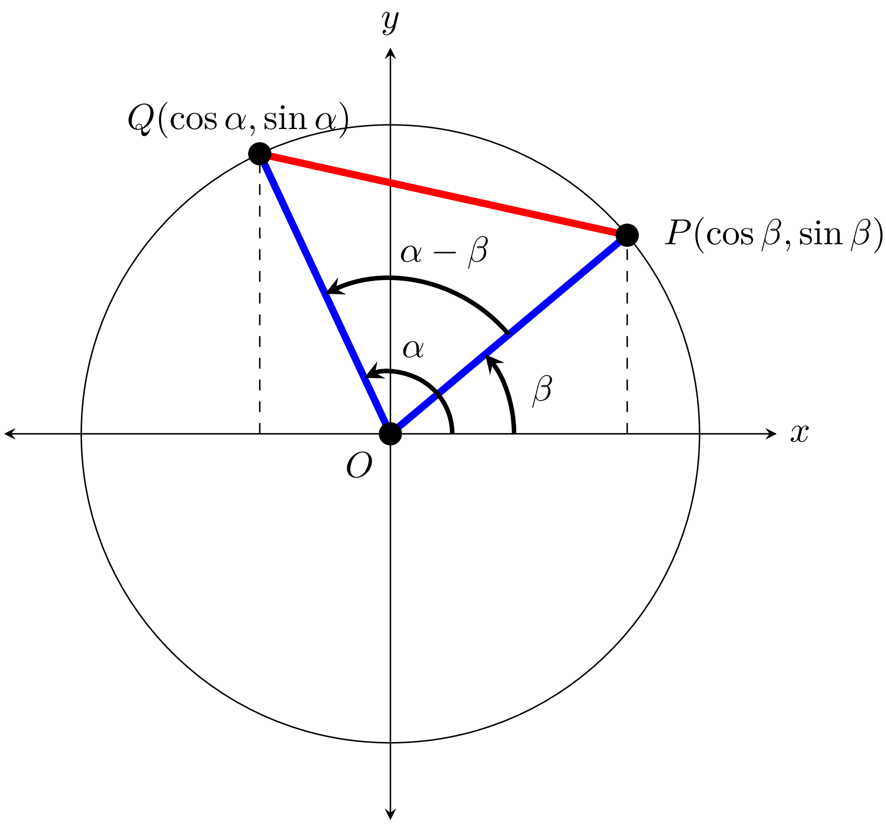
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$\Diamond$

Proof. First we will prove the Subtraction Formula for Cosine

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

We begin by considering two points on the unit circle. Point P is at an angle of β in standard position with coordinates $(\cos\beta, \sin\beta)$ and point Q is at an angle of α in standard position with coordinates $(\cos\alpha, \sin\alpha)$.



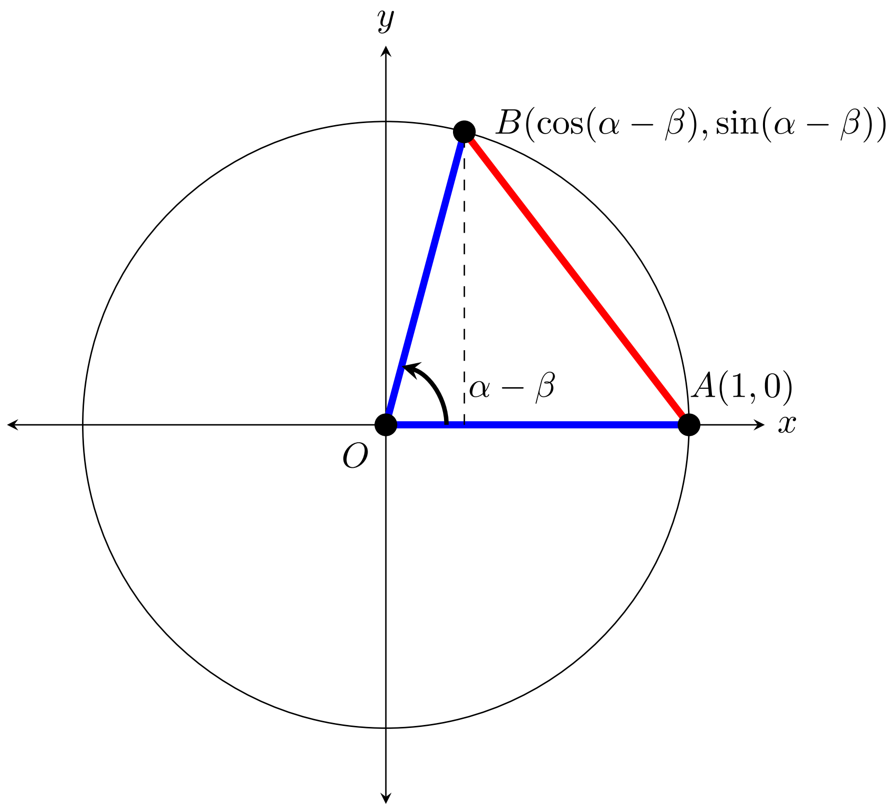
We use the Distance Formula to calculate the distance between P and Q to get

$$\begin{aligned} d(P,Q) &= \sqrt{(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2} \\ &= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} \\ &= \sqrt{(\cos^2\alpha+\sin^2\alpha) + (\cos^2\beta+\sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}. \end{aligned}$$

By the Pythagorean Identity (Definition 1.5.20), $\cos^2\alpha + \sin^2\alpha = 1$ and $\cos^2\beta + \sin^2\beta = 1$. Thus the distance becomes

$$\begin{aligned} d(P,Q) &= \sqrt{1+1-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{2-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}. \end{aligned}$$

Next, consider two additional points on a second unit circle. Point A has coordinates $(1, 0)$ and Point B is at an angle of $\alpha - \beta$ in standard position with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$.



The distance between A and B is

$$\begin{aligned} d(A,B) &= \sqrt{(\cos(\alpha-\beta)-1)^2 + (\sin(\alpha-\beta)-0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} \\ &= \sqrt{(\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)) - 2\cos(\alpha-\beta) + 1} \\ &= \sqrt{1 - 2\cos(\alpha-\beta) + 1} \\ &= \sqrt{2 - 2\cos(\alpha-\beta)}. \end{aligned}$$

Note that since OP, OQ, OA, and OB are line segments from the center to points on the unit circle, they are congruent and have length of 1. Also note that $\angle POQ = \angle AOB = \alpha - \beta$. Since two sides and the included angle of $\triangle OPQ$ and $\triangle OAB$ are congruent, we can conclude by the Side-Angle-Side Theorem (SAS) in geometry that the two triangles are congruent. Thus, corresponding sides have the same lengths, giving us $d(P,Q) = d(A,B)$. Substituting our results for $d(P,Q)$ and $d(A,B)$ we get

$$\begin{aligned} d(P,Q) &= d(A,B) \\ \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} &= \sqrt{2 - 2\cos(\alpha-\beta)} \\ 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 2 - 2\cos(\alpha-\beta) \\ -2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= -2\cos(\alpha-\beta). \end{aligned}$$

Dividing both sides by -2 we arrive at the Subtraction Formula for Cosine

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta).$$

To prove the Addition Formula for Cosine, replace β with $-\beta$ in the Subtraction Formula and use the Even and Odd Trigonometric Properties (Definition 1.5.22) where $\sin(-\beta) = -\sin\beta$ and $\cos(-\beta) = \cos\beta$ to get

$$\begin{aligned} \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) &= \cos(\alpha - (-\beta)) \\ \cos\alpha\cos(\beta) + \sin\alpha(-\sin\beta) &= \cos(\alpha+\beta) \\ \cos\alpha\cos\beta - \sin\alpha\sin\beta &= \cos(\alpha+\beta). \end{aligned}$$

■

Example 3.2.2 Find the exact value of $\cos 105°$.

Solution. First note that $105° = 60° + 45°$. Now, using the Addition Formula for Cosine (Definition 3.2.1),

$$\begin{aligned} \cos 105° &= \cos(60° + 45°) \\ &= \cos 60° \cos 45° - \sin 60° \sin 45° \\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ &= \frac{1}{4}\left(\sqrt{2} - \sqrt{6}\right). \end{aligned}$$

□

Example 3.2.3 Find the exact value of $\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.

Solution. Using the Subtraction Formula for Cosine we get

$$\begin{aligned} \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right). \end{aligned}$$

□

Example 3.2.4 Find the exact value of the expression $\cos 25° \cos 35° - \sin 25° \sin 35°$.

Solution. Notice this expression is the Addition Formula for Cosine with $\alpha = 25°$ and $\beta = 35°$. So

$$\cos 25° \cos 35° - \sin 25° \sin 35° = \cos(25° + 35°) = \cos 60° = \frac{1}{2}.$$

□

3.2.2 Addition and Subtraction Formulas for Sine

Next we will learn about the addition and subtraction formulas for the sine function.

Definition 3.2.5 Addition and Subtraction Formulas for Sine.

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}$$

◊

We will prove the Addition Formula for Sine in Example 3.2.10 and the Subtraction Formula can be established using the Even and Odd Properties

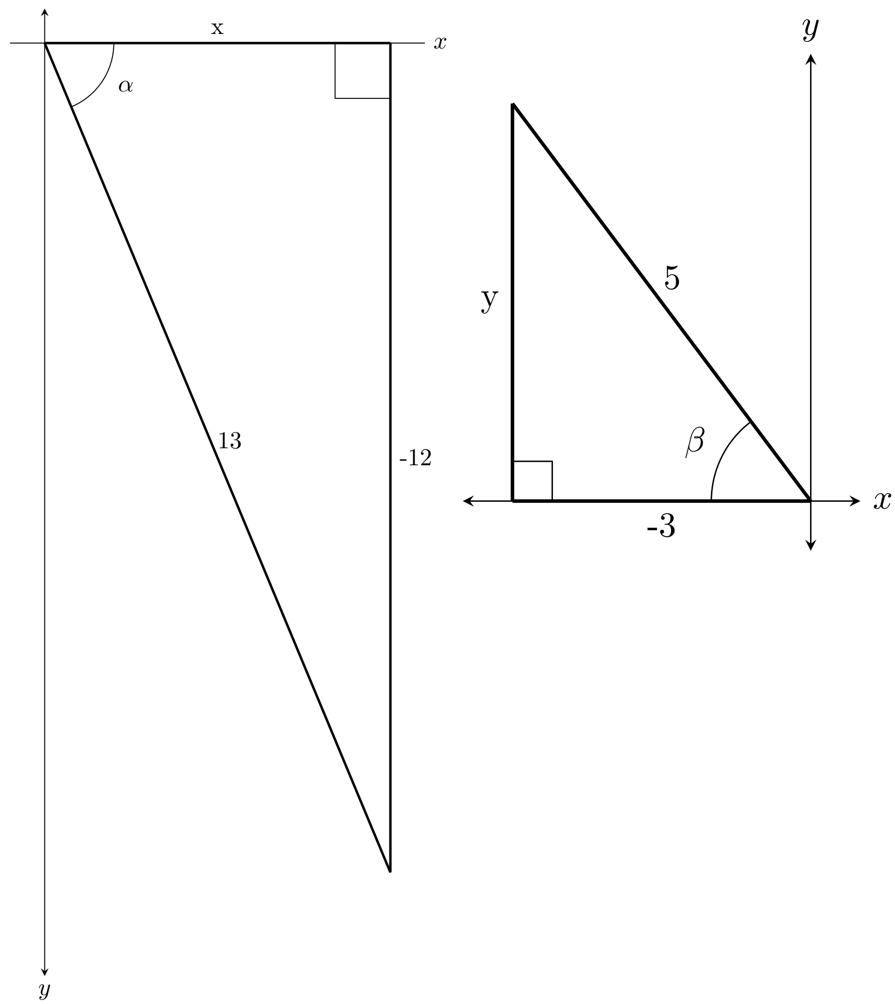
(Definition 1.5.22).

Example 3.2.6 Given $\sin \alpha = -\frac{12}{13}$, with $\frac{3\pi}{2} < \alpha < 2\pi$ and $\cos \beta = -\frac{3}{5}$, with $\frac{\pi}{2} < \beta < \pi$, find the exact value of $\sin(\alpha + \beta)$.
Solution. The Addition Formula for Sine gives us

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

At this moment, we do not know the exact values of $\cos \alpha$ and $\sin \beta$ but we can compute them.

Given $\sin \alpha = -\frac{12}{13}$ with $\frac{3\pi}{2} < \alpha < 2\pi$ and $\cos \beta = -\frac{3}{5}$ with $\frac{\pi}{2} < \beta < \pi$, we can draw the following triangles associated with α and β, respectively:



Next, using the Pythagorean Theorem, we solve for the missing sides on the triangles:

$$\begin{aligned} x^2 + (-12)^2 &= 13^2 \\ x + 144 &= 169 \\ x^2 &= 25 \\ x &= 5. \end{aligned}$$

Thus we get

$$\cos \alpha = \frac{5}{13}.$$

Similarly,

$$\begin{aligned}(-3)^2 + y^2 &= 5^2\\ 9 + y^2 &= 25\\ y^2 &= 16\\ y &= 4.\end{aligned}$$

Thus we get

$$\sin\beta = \frac{4}{5}.$$

We now have all the information needed to proceed.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)\\ &= \frac{36}{65} + \frac{20}{65}\\ &= \frac{56}{65}.\end{aligned}$$

Notice we did not have to know the values of α or β to do this example. □

3.2.3 Addition and Subtraction Formulas for Tangent

Now we will learn about the addition and subtraction formulas for the tangent function.

Definition 3.2.7 Addition and Subtraction Formulas for Tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\ \tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\end{aligned}$$

◊

Proof. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ as long as $\cos\theta \neq 0$. Using this fact and our new formulas for the sum of sine and cosine, we get

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\\ &= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\dfrac{\sin\beta}{\cos\beta}}\end{aligned}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

The subtraction formula for tangent can be established using the Even and Odd Properties (Definition 1.5.22). ■

Example 3.2.8 Find the exact value of $\tan\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

Solution. Using the Addition Formula for Tangent (Definition 3.2.7), we get

$$\begin{aligned}
\tan\left(\frac{3\pi}{4} + \frac{\pi}{6}\right) &= \frac{\tan\frac{3\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{3\pi}{4}\tan\frac{\pi}{6}} \\
&= \frac{(-1) + \frac{\sqrt{3}}{3}}{1 - (-1)\cdot\frac{\sqrt{3}}{3}} \\
&= \frac{-1 + \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} \\
&= \frac{\frac{-3+\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}} \\
&= \frac{-3 + \sqrt{3}}{3 + \sqrt{3}}.
\end{aligned}$$

□

3.2.4 Cofunction Identities

Recall the Cofunction Identities (Definition 1.4.7):

$$\begin{aligned}
\sin\theta &= \cos\left(\frac{\pi}{2} - \theta\right), & \cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right) \\
\tan\theta &= \cot\left(\frac{\pi}{2} - \theta\right), & \cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right) \\
\sec\theta &= \csc\left(\frac{\pi}{2} - \theta\right), & \csc\theta &= \sec\left(\frac{\pi}{2} - \theta\right)
\end{aligned}$$

Armed with the knowledge of the subtraction formulas, we can prove the Cofunction Identities.

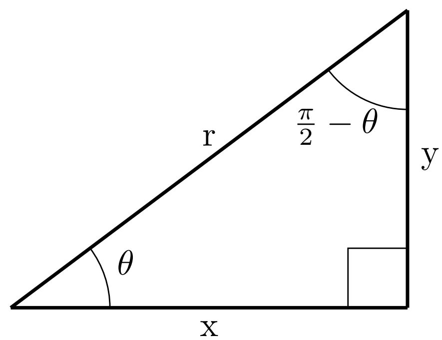
Proof. We will prove the Cofunction Identity for $\sin\theta$ in Example 3.2.9. The proof for $\cos\theta$ is given as Exercise 3.2.7.97. The Cofunction Identities for $\tan\theta$ and $\cot\theta$ can be found using the Quotient Identities (Definition 1.4.3); for $\csc\theta$ and $\sec\theta$ can be found using the Reciprocal Identities (Definition 1.4.2). ■

Example 3.2.9 Use the Subtraction Formula for Sine to establish the identity $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

Solution. To establish an identity, we will start from one side of the equation and use properties to end up with the expression on the other side of the equation. So,

$$\begin{aligned}
\sin\left(\frac{\pi}{2} - \theta\right) &= \sin\left(\frac{\pi}{2}\right)\cos(\theta) - \cos\left(\frac{\pi}{2}\right)\sin(\theta) \\
&= 1\cdot\cos(\theta) - 0\cdot\sin(\theta) \\
&= \cos(\theta).
\end{aligned}$$

Visually, we have $\cos\theta = \frac{x}{r} = \sin\left(\frac{\pi}{2} - \theta\right)$.



□

Example 3.2.10 Prove the Addition Formula for Sine

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Solution. We begin by using the cofunction identity

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right).\end{aligned}$$

By the Subtraction Formula for Cosine:

$$\begin{aligned}= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right) &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\end{aligned}$$

in the last step, we use the Cofunction Identity. □

3.2.5 Sums of Sines and Cosines

Sometimes we may come across functions of the form

$$a\sin x + b\cos x.$$

It can often be useful to rewrite this expression as a single trigonometric function.

Definition 3.2.11 For any real numbers a and b, let θ be an angle in standard position where $P(a, b)$ is a point on the terminal side of θ. Then

$$a\sin x + b\cos x = \sqrt{a^2 + b^2}\sin(x + \theta).$$

◊

Proof. We begin by considering the triangle formed by angle θ and point $P(a, b)$, shown in Figure 3.2.12. By the Pythagorean Theorem, the hypotenuse of this triangle, with base a and height b, is $\sqrt{a^2 + b^2}$. According to Definition 1.4.1, we have

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

or equivalently:

$$a = \sqrt{a^2 + b^2}\cos\theta, \quad b = \sqrt{a^2 + b^2}\sin\theta.$$

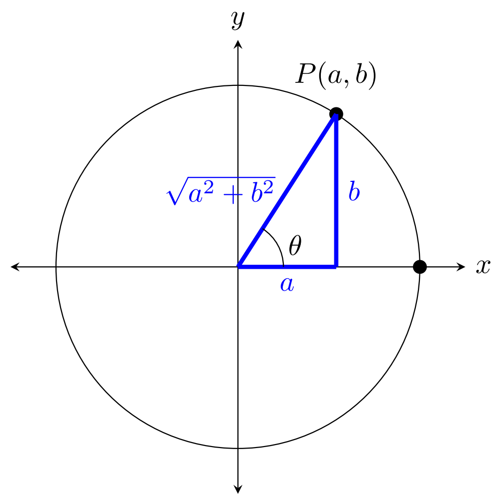


Figure 3.2.12 A triangle is formed by angle θ and point $P(a,b)$.

Therefore, using the addition formula for sine, we get

$$\begin{aligned} a\sin x + b\cos x &= \sqrt{a^2+b^2}\cos\theta\sin x + \sqrt{a^2+b^2}\sin\theta\cos x \\ &= \sqrt{a^2+b^2}\,(\cos\theta\sin x + \sin\theta\cos x) \\ &= \sqrt{a^2+b^2}\sin(x+\theta). \end{aligned}$$

■

Example 3.2.13 Express

$$-\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

in terms of sine only.

Solution. To express the given expression in terms of sine only, we will use Definition 3.2.11. Considering the point $P(a,b) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, which lies in Quadrant II, we determine the angle θ. Using either Table 1.5.18 or inverse trigonometric methods where

$$\tan\theta = \frac{b}{a} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}},$$

we find

$$\theta = 150^\circ.$$

Therefore, by Definition 3.2.11, we have:

$$\begin{aligned} -\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x &= \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\sin(x+150^\circ) \\ &= \sqrt{\frac{3}{4}+\frac{1}{4}}\sin(x+150^\circ) \\ &= \sin(x+150^\circ). \end{aligned}$$

□

3.2.6 Summary

To review, the addition and subtraction formulas are:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.
\end{aligned}$$

3.2.7 Exercises

Exercise Group. Use the Addition or Subtraction Formula to find the exact value of each expression.

1. $\cos(15^\circ)$

Answer. $\frac{\sqrt{6}+\sqrt{2}}{4}$

2. $\sin(75^\circ)$

Answer. $\frac{\sqrt{6}+\sqrt{2}}{4}$

3. $\sin(195^\circ)$

Answer. $\frac{-\sqrt{6}+\sqrt{2}}{4}$

4. $\tan(165^\circ)$

Answer. $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

5. $\tan(105^\circ)$

Answer. $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

6. $\cos(255^\circ)$

Answer. $\frac{-\sqrt{6}+\sqrt{2}}{4}$

7. $\tan\left(\frac{5\pi}{12}\right)$

Hint. $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$

Answer. $\frac{3+\sqrt{3}}{3-\sqrt{3}}$

8. $\sin\left(\frac{7\pi}{12}\right)$

Hint. $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

Answer. $\frac{\sqrt{6}+\sqrt{2}}{4}$

9. $\cos\left(\frac{\pi}{12}\right)$

Hint. $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

Answer. $\frac{\sqrt{6}+\sqrt{2}}{4}$

10. $\sin\left(\frac{13\pi}{12}\right)$

Hint. $\frac{13\pi}{12} = \frac{11\pi}{6} - \frac{3\pi}{4}$

Answer. $\frac{\sqrt{2}-\sqrt{6}}{4}$

11. $\cos\left(\frac{17\pi}{12}\right)$

Hint. $\frac{17\pi}{12} = \frac{5\pi}{3} - \frac{\pi}{4}$

Answer. $\frac{\sqrt{2}-\sqrt{6}}{4}$

12. $\tan\left(-\frac{\pi}{12}\right)$

Hint. $-\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{3}$

Answer. $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

Exercise Group. Use the Addition or Subtraction Formula to find the exact value of each expression.

13. $\sin(172^\circ)\cos(68^\circ) + \cos(172^\circ)\sin(68^\circ)$

Answer. $-\frac{\sqrt{3}}{2}$

14. $\sin(317^\circ)\cos(257^\circ) - \cos(317^\circ)\sin(257^\circ)$

Answer. $\frac{\sqrt{3}}{2}$

15. $\cos(337^\circ)\cos(22^\circ) + \sin(337^\circ)\sin(22^\circ)$

Answer. $\frac{1}{\sqrt{2}}$

16. $\cos(59^\circ)\cos(211^\circ) - \sin(59^\circ)\sin(211^\circ)$

Answer. 0

17. $\frac{\tan(85°)-\tan(25°)}{1+\tan(85°)\tan(25°)}$

Answer. $\sqrt{3}$

18. $\sin\left(\frac{5\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)-\cos\left(\frac{5\pi}{16}\right)\sin\left(\frac{\pi}{16}\right)$

Answer. $\frac{\sqrt{2}}{2}$

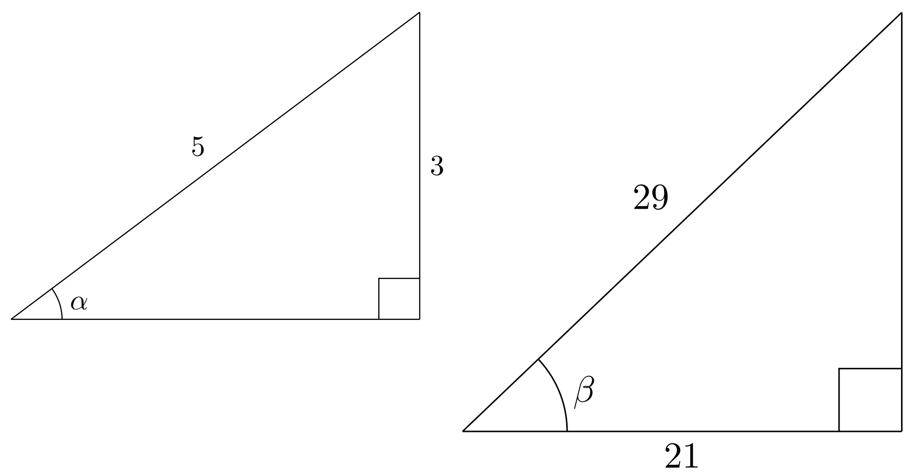
19. $\cos\left(\frac{5\pi}{22}\right)\cos\left(\frac{3\pi}{11}\right)-\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{3\pi}{11}\right)$

Answer. 0

20. $\frac{\tan\left(\frac{\pi}{7}\right)+\tan\left(\frac{4\pi}{21}\right)}{1-\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{4\pi}{21}\right)}$

Answer. $\sqrt{3}$

Exercise Group. Find the exact value of each expression given:



21. $\sin(\alpha)$

Answer. $\frac{3}{5}$

22. $\cos(\alpha)$

Answer. $\frac{4}{5}$

23. $\tan(\alpha)$

Answer. $\frac{3}{4}$

24. $\sin(\beta)$

Answer. $\frac{20}{29}$

25. $\cos(\beta)$

Answer. $\frac{21}{29}$

26. $\tan(\beta)$

Answer. $\frac{20}{21}$

27. $\sin(\alpha+\beta)$

Answer. $\frac{3}{5}\cdot\frac{21}{29}+\frac{4}{5}\cdot\frac{20}{29}=\frac{143}{145}$

28. $\cos(\alpha+\beta)$

Answer. $\frac{4}{5}\cdot\frac{21}{29}-\frac{3}{5}\cdot\frac{20}{29}=\frac{24}{145}$

29. $\tan(\alpha+\beta)$

Answer. $\frac{\frac{3}{4}+\frac{20}{21}}{1-\frac{3}{4}\cdot\frac{20}{21}}=\frac{143}{24}$

30. $\sin(\alpha-\beta)$

Answer. $\frac{3}{5}\cdot\frac{21}{29}-\frac{4}{5}\cdot\frac{20}{29}=-\frac{17}{145}$

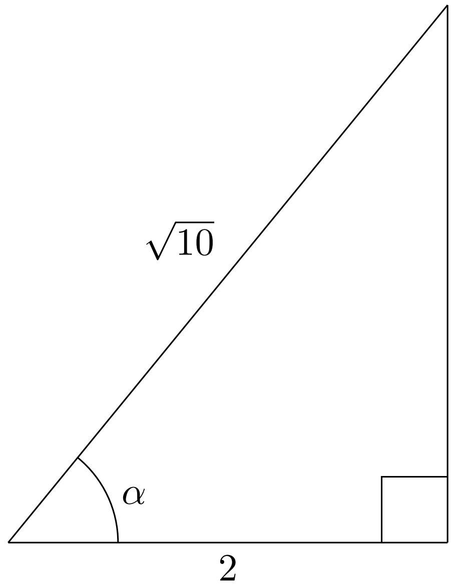
31. $\cos(\alpha-\beta)$

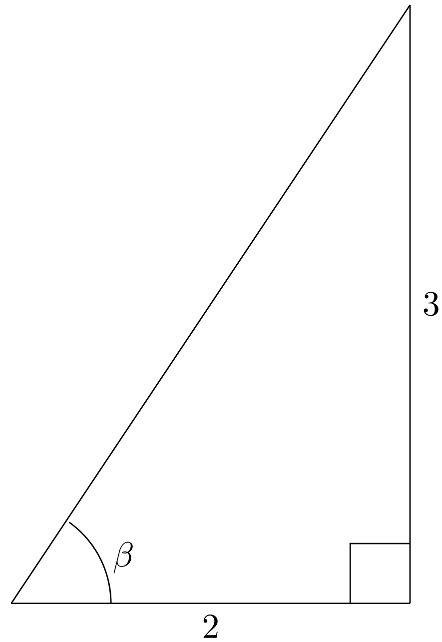
Answer. $\frac{4}{5}\cdot\frac{21}{29}+\frac{3}{5}\cdot\frac{20}{29}=\frac{144}{145}$

32. $\tan(\alpha-\beta)$

Answer. $\frac{\frac{3}{4}-\frac{20}{21}}{1+\frac{3}{4}\cdot\frac{20}{21}}=-\frac{17}{24}$

Exercise Group. Find the exact value of each expression given:





33. $\sin(\alpha)$

Answer. $\frac{\sqrt{6}}{\sqrt{10}} = \sqrt{\frac{3}{5}}$

34. $\cos(\alpha)$

Answer. $\frac{2}{\sqrt{10}}$

35. $\tan(\alpha)$

Answer. $\frac{\sqrt{6}}{2}$

36. $\sin(\beta)$

Answer. $\frac{3}{\sqrt{13}}$

37. $\cos(\beta)$

Answer. $\frac{2}{\sqrt{13}}$

38. $\tan(\beta)$

Answer. $\frac{3}{2}$

39. $\sin(\alpha + \beta)$

Answer. $\frac{\sqrt{6}}{\sqrt{10}} \cdot \frac{2}{\sqrt{13}} + \frac{2}{\sqrt{10}} \cdot \frac{3}{\sqrt{13}} = \frac{2\sqrt{6}+6}{\sqrt{130}}$

40. $\cos(\alpha + \beta)$

Answer. $\frac{2}{\sqrt{10}} \cdot \frac{2}{\sqrt{13}} - \frac{\sqrt{6}}{\sqrt{10}} \cdot \frac{3}{\sqrt{13}} = \frac{4-3\sqrt{6}}{\sqrt{130}}$

41. $\tan(\alpha + \beta)$

Answer. $\frac{\frac{\sqrt{6}}{2}+\frac{3}{2}}{1-\frac{\sqrt{6}}{2}\cdot\frac{3}{2}} = \frac{2\sqrt{6}+6}{4-3\sqrt{6}}$

42. $\sin(\alpha - \beta)$

Answer. $\frac{\sqrt{6}}{\sqrt{10}} \cdot \frac{2}{\sqrt{13}} - \frac{2}{\sqrt{10}} \cdot \frac{3}{\sqrt{13}} = \frac{2\sqrt{6}-6}{\sqrt{130}}$

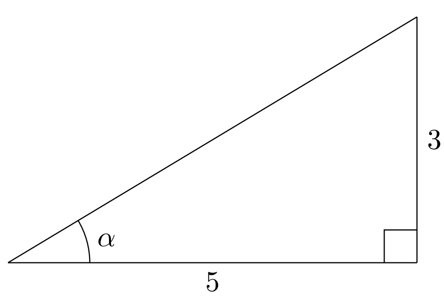
43. $\cos(\alpha - \beta)$

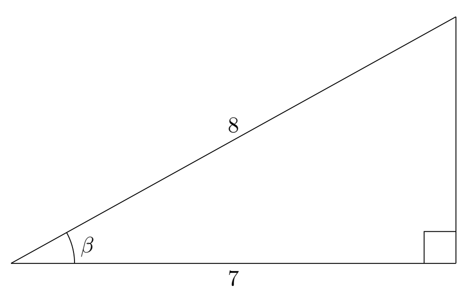
Answer. $\frac{2}{\sqrt{10}} \cdot \frac{2}{\sqrt{13}} + \frac{\sqrt{6}}{\sqrt{10}} \cdot \frac{3}{\sqrt{13}} = \frac{4+3\sqrt{6}}{\sqrt{130}}$

44. $\tan(\alpha - \beta)$

Answer. $\frac{\frac{\sqrt{6}}{2}-\frac{3}{2}}{1+\frac{\sqrt{6}}{2}\cdot\frac{3}{2}} = \frac{2\sqrt{6}-6}{4+3\sqrt{6}}$

Exercise Group. Find the exact value of each expression given:





45. $\sin(\alpha)$

Answer. $\frac{3}{\sqrt{34}}$

46. $\cos(\alpha)$

Answer. $\frac{5}{\sqrt{34}}$

47. $\tan(\alpha)$

Answer. $\frac{3}{5}$

48. $\sin(\beta)$

Answer. $\frac{\sqrt{15}}{8}$

49. $\cos(\beta)$

Answer. $\frac{7}{8}$

50. $\tan(\beta)$

Answer. $\frac{\sqrt{15}}{7}$

51. $\sin(\alpha+\beta)$

Answer. $\frac{3}{\sqrt{34}}\cdot\frac{7}{8}+\frac{5}{\sqrt{34}}\cdot\frac{\sqrt{15}}{8}=\frac{21+5\sqrt{15}}{8\sqrt{34}}$

52. $\cos(\alpha+\beta)$

Answer. $\frac{5}{\sqrt{34}}\cdot\frac{7}{8}-\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{15}}{8}=\frac{35-3\sqrt{15}}{8\sqrt{34}}$

53. $\tan(\alpha+\beta)$

Answer. $\frac{\frac{3}{5}+\frac{\sqrt{15}}{7}}{1-\frac{3}{5}\cdot\frac{\sqrt{15}}{7}}=\frac{21+5\sqrt{15}}{35-3\sqrt{15}}$

54. $\sin(\alpha-\beta)$

Answer. $\frac{3}{\sqrt{34}}\cdot\frac{7}{8}-\frac{5}{\sqrt{34}}\cdot\frac{\sqrt{15}}{8}=\frac{21-5\sqrt{15}}{8\sqrt{34}}$

55. $\cos(\alpha-\beta)$

Answer. $\frac{5}{\sqrt{34}}\cdot\frac{7}{8}+\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{15}}{8}=\frac{35+3\sqrt{15}}{8\sqrt{34}}$

56. $\tan(\alpha-\beta)$

Answer. $\frac{\frac{3}{5}-\frac{\sqrt{15}}{7}}{1+\frac{3}{5}\cdot\frac{\sqrt{15}}{7}}=\frac{21-5\sqrt{15}}{35+3\sqrt{15}}$

Exercise Group. Find the exact of each expression given $\sin\alpha=\frac{20}{29}$, $0<\alpha<\frac{\pi}{2}$ and $\cos\beta=\frac{24}{25}$, $0<\beta<\frac{\pi}{2}$

57. $\cos(\alpha)$

Answer. $\frac{21}{29}$

58. $\tan(\alpha)$

Answer. $\frac{20}{21}$

59. $\sin(\beta)$

Answer. $\frac{7}{25}$

60. $\tan(\beta)$

Answer. $\frac{7}{24}$

61. $\sin(\alpha+\beta)$

Answer. $\frac{627}{725}$

62. $\sin(\alpha-\beta)$

Answer. $\frac{333}{725}$

63. $\cos(\alpha+\beta)$

Answer. $\frac{364}{725}$

64. $\cos(\alpha-\beta)$

Answer. $\frac{644}{725}$

65. $\tan(\alpha+\beta)$

Answer. $\frac{627}{364}$

66. $\tan(\alpha-\beta)$

Answer. $\frac{333}{644}$

Exercise Group. Find the exact value of each expression given $\tan\alpha=\frac{8}{15}$, $\pi<\alpha<\frac{3\pi}{2}$ and $\cos\beta=-\frac{3}{5}$, $\frac{\pi}{2}<\beta<\pi$

67. $\sin(\alpha)$

Answer. $-\frac{8}{17}$

68. $\cos(\alpha)$

Answer. $-\frac{15}{17}$

69. $\sin(\beta)$

Answer. $\frac{4}{5}$

70. $\tan(\beta)$

Answer. $-\frac{4}{3}$

71. $\sin(\alpha+\beta)$

Answer. $-\frac{36}{85}$

72. $\sin(\alpha-\beta)$

Answer. $\frac{84}{85}$

73. $\cos(\alpha+\beta)$

Answer. $\frac{77}{85}$

74. $\cos(\alpha-\beta)$

Answer. $\frac{13}{85}$

75. $\tan(\alpha+\beta)$

Answer. $-\frac{36}{77}$

76. $\tan(\alpha-\beta)$

Answer. $\frac{84}{13}$

Exercise Group. Verify the identity.

77. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

78. $\sin(\theta - \pi) = -\sin\theta$

79. $\cos(\theta - \pi) = -\cos\theta$

80. $\tan(\theta - \pi) = \tan\theta$

81. $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$

82. $\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right)$

83. $\cos\left(\theta + \frac{\pi}{3}\right) = -\sin\left(\theta - \frac{\pi}{6}\right)$

84. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

85. $\frac{\sin(x+y)}{\cos x \cos y} = \tan x + \tan y$

86. $\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

87. $\sin(x+y) - \sin(x-y) = 2\cos x \sin y$

88. $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$

Exercise Group. Write each expression in terms on sine only. Round your angles to one decimal.

89. $-\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$

Answer. $\sin(x + 225°)$

90. $\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$

Answer. $\sin(x + 330°)$

91. $\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$

Answer. $\sin(x + 60°)$

92. $-\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$

Answer. $\sin(x + 210°)$

93. $3\sin x + 7\cos x$

Answer. $\sqrt{58}\sin(x + 66.8°)$

94. $-5\sin x - 9\cos x$

Answer. $\sqrt{106}\sin(x + 240.9°)$

95. $8\sin x - 2\cos x$

Answer. $\sqrt{68}\sin(x + 346.0°)$

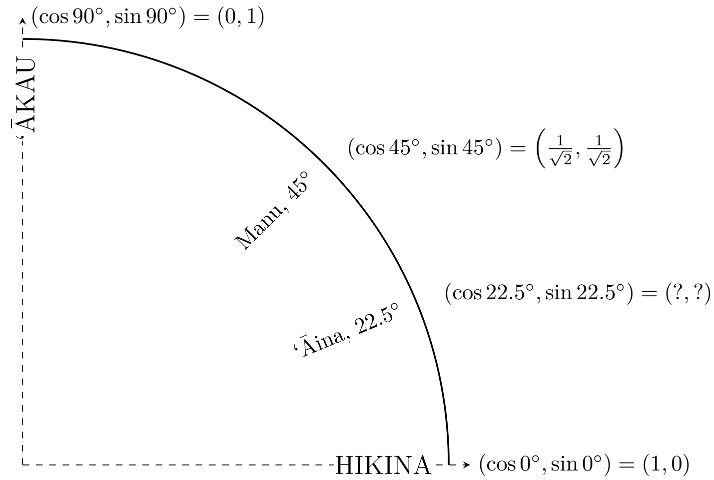
96. $-7\sin x + 4\cos x$

Answer. $\sqrt{65}\sin(x + 150.3°)$

97. Use the Subtraction Formula for Cosine to prove the Cofunction Identity for Sine: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.

3.3 Double-Angle and Half-Angle Formulas

Suppose we want to accurately position the Hawaiian Star Compass on the Unit Circle. In Figure 1.1.4, the house for Manu is located halfway between Hikina and ʻĀkau, resulting in an angle of 45°. By applying right triangle trigonometry, we can determine the exact coordinates of Manu as $(\cos(45°), \sin(45°)) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. However, as we move to the house of ʻĀina, located halfway between Manu and Hikina, we encounter a problem. The angle for ʻĀina is 22.5°, which is not explicitly listed in Table 1.5.18. Therefore, we must resort to a calculator for numerical approximations.



In this section, we will learn about the double and half-angle formulas for trigonometry. These formulas allow us to determine exact trigonometric function values for angles that are double or half of the common angles. This will enable us to use our existing knowledge of trigonometric functions at 45° and apply the half-angle formulas to obtain exact values at 22.5°.

3.3.1 Double-Angle Formulas

Recall the addition formula for sine:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{3.3.1}$$

Consider the case when the two angles are equal. We will call this angle θ, so let $\alpha = \theta$ and $\beta = \theta$. Then Eq (3.3.1) becomes

$$\begin{aligned} \sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta. \end{aligned}$$

Thus we obtain a formula for sine of twice the angle θ.

Definition 3.3.1 Double-Angle Formulas.

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \cos 2\theta &= 1 - 2\sin^2\theta \\ \cos 2\theta &= 2\cos^2\theta - 1 \\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta} \end{aligned}$$

◊

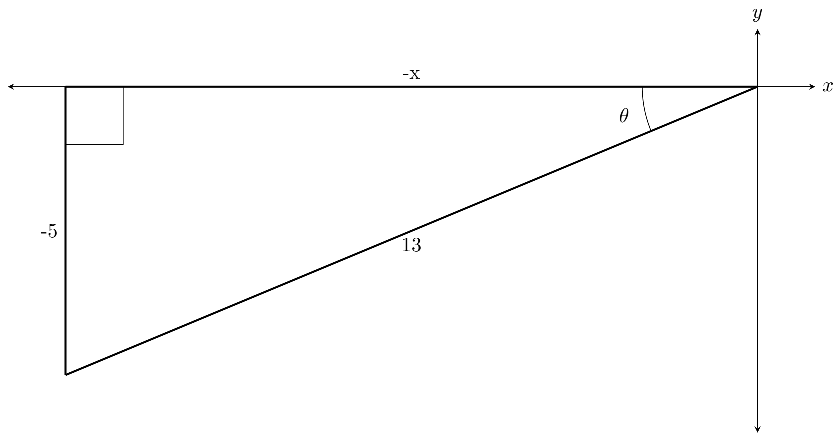
The proofs for the Double-Angle Formulas for Cosine and Tangent are left as exercises (Exercise 3.3.4.6-Exercise 3.3.4.9).

Remark 3.3.2 Notice that there are three variations of the double-angle formula for cosine. All three equations give the correct answer, however, one version may be more convenient depending on the given information. For example, if we are given the value of $\sin\theta$, it may be easier to select the version that solely involves $\sin\theta$ and does not include $\cos\theta$.

Example 3.3.3 Given $\sin\theta = -\frac{5}{13}$ and θ lies in Quadrant III, find the exact value of:

(a) $\sin(2\theta)$

Solution. By the double-angle formula, we have $\sin(2\theta) = 2\sin\theta\cos\theta$. We are given the value of $\sin\theta$, but we do not have $\cos\theta$. To find $\cos\theta$, we will draw the triangle formed from $\sin\theta = -\frac{5}{13}$ where θ lies in Quadrant III.



Using the Pythagorean Theorem, we can solve the triangle:

$$\begin{aligned} x^2 + (-5)^2 &= 13^2 \\ x^2 + 25 &= 169 \\ x^2 &= 144 \\ x &= 12. \end{aligned}$$

Thus we have

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = -\frac{12}{13}$$

and

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}.$$

With this new information, we can use the double-angle formula to find $\sin(2\theta)$:

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{5}{13}\right)\left(-\frac{12}{13}\right) = \frac{120}{169}.$$

(b) $\cos(2\theta)$

Solution. To compute $\cos 2\theta$, notice there are three different formulas: $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$, or $\cos 2\theta = 2\cos^2\theta - 1$. Using any of the three equations will give us the correct answer. However, given that we know $\sin\theta = -\frac{5}{13}$, it may be easier to use $\cos 2\theta = 1 - 2\sin^2\theta$, since the other two equations require us to know $\cos\theta$.

Without having to draw the triangle, we get

$$\begin{aligned}\cos 2\theta &= 1 - 2\sin^2\theta \\ &= 1 - 2\left(-\frac{5}{13}\right)^2 \\ &= 1 - 2\left(\frac{25}{169}\right) \\ &= \frac{169}{169} - \frac{50}{169} \\ &= \frac{119}{169}.\end{aligned}$$

(c) $\tan(2\theta)$

Solution. Using the double-angle formula for tangent, we get

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2\left(\frac{5}{12}\right)}{1 - \left(\frac{5}{12}\right)^2} \\ &= \frac{\frac{10}{12}}{1 - \frac{25}{144}} \\ &= \frac{\frac{10}{12}}{\frac{119}{144}} \\ &= \frac{10}{12} \cdot \frac{144}{119} \\ &= \frac{120}{119}.\end{aligned}$$

□

Example 3.3.4 Write $\sin(3\theta)$ in terms of $\sin\theta$.

Solution.

$$\begin{aligned}\sin(3\theta) &= \sin(2\theta + \theta) \\ &= \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta && \text{addition formula} \\ &= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta && \text{double-angle formula} \\ &= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta && \text{Pythagorean Identity} \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

□

3.3.2 Reducing Powers Formulas

The double-angle formula for cosine expresses a trigonometric function in terms of the square of another trigonometric function. By rearranging the terms, we can derive formulas for reducing the powers of sine, cosine, and tangent in expressions with even powers to terms involving only cosine. These formulas are particularly useful in calculus.

Definition 3.3.5 Formulas for Reducing Powers.

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

◊

Proof. To prove the first formula, solve for $\sin^2\theta$ in the double-angle formula: $\cos 2\theta = 1 - 2\sin^2\theta$. The second formula is obtained similarly by solving for $\cos^2\theta$ in the formula $\cos 2\theta = 2\cos^2\theta - 1$. The first two formulas can be used to obtain the third formula:

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\dfrac{1-\cos(2\theta)}{2}}{\dfrac{1+\cos(2\theta)}{2}} = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}.$$

■

Example 3.3.6 Write $\sin^4\theta$ as an expression that does not involve powers of sine or cosine greater than 1.

Solution. We will use the Reducing Powers Formula twice.

$$\begin{aligned}
\sin^4\theta &= (\sin^2\theta)^2 \\
&= \left(\frac{1-\cos(2\theta)}{2}\right)^2 && \text{reducing powers} \\
&= \frac{1}{4}\left(1 - 2\cos(2\theta) + \cos^2(2\theta)\right) \\
&= \frac{1}{4}\left(1 - 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right) && \text{reducing powers} \\
&= \frac{1}{4}\left(1 - 2\cos(2\theta) + \frac{1}{2} + \frac{\cos(4\theta)}{2}\right) \\
&= \frac{1}{4}\left(\frac{3}{2} - \frac{4}{2}\cos(2\theta) + \frac{1}{2}\cos(4\theta)\right) \\
&= \frac{1}{8}\left(3 - 4\cos(2\theta) + \cos(4\theta)\right).
\end{aligned}$$

□

3.3.3 Half-Angle Formulas

Another set of useful formulas are the half-angle formulas.

Definition 3.3.7 Half-Angle Formulas.

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}, \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}, \qquad \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

The choice of the + or - sign depends on the Quadrant in which $\theta/2$ lies. ◊

Proof. We take the square root of both sides of the Formulas for Reducing Powers (Definition 3.3.5) and halve the angle (θ becomes $\frac{\theta}{2}$ and 2θ becomes θ) to arrive at our formulas. ■

Example 3.3.8 Locating ʻĀina. We are now ready to revisit the problem posed at the start of this section where we were asked to determine the exact coordinates of the house ʻĀina on the Unit Circle.

Solution. We know the coordinates are at

$$(\cos 22.5°, \sin 22.5°).$$

To determine the exact value of $\cos 22.5°$, we use the half-angle formula along with the known value $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$:

$$\begin{aligned}
\cos 22.5° &= \cos\left(\frac{45}{2}\right)^{\circ} \\
&= \sqrt{\frac{1+\cos 45°}{2}} \\
&= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{4}} \\
&= \sqrt{\frac{2+\sqrt{2}}{4}} \\
&= \frac{1}{2}\sqrt{2+\sqrt{2}}.
\end{aligned}$$

Since the half-angle formula has $\pm$, we check the quadrant. In this case, our angle is $22.5°$, which is in Quadrant I. Therefore, we choose the positive value.

Finding the exact value of $\sin 22.5°$ is left for Exercise 3.3.4.1. □

Example 3.3.9 Given $\sin\theta = -\frac{5}{13}$ and θ lies in Quadrant III, find the exact value of:

(a) $\sin\frac{\theta}{2}$

Solution. Notice the Half-Angle Formulas all require us to know $\cos\theta$. Since the given information describes the same triangle as in Example 3.3.3, we refer to that problem to get $\cos\theta = -\frac{12}{13}$.

Next, since θ is in Quadrant III, $180° < \theta < 270°$, dividing by 2 gives us $\frac{180°}{2} < \frac{\theta}{2} < \frac{270°}{2}$ or $90° < \frac{\theta}{2} < 135°$. Therefore, we conclude that $\frac{\theta}{2}$ lies in Quadrant II.

To calculate $\sin\frac{\theta}{2}$, we first note that because $\frac{\theta}{2}$ lies in Quadrant II, $\sin\frac{\theta}{2} > 0$ so we will choose the positive (+) sign in the Half-Angle Formula:

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\begin{aligned}
&= \sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}} \\
&= \sqrt{\frac{1+\frac{12}{13}}{2}} \\
&= \sqrt{\frac{\frac{13}{13}+\frac{12}{13}}{2}} \\
&= \sqrt{\frac{\frac{25}{13}}{2}} \\
&= \sqrt{\frac{25}{26}}.
\end{aligned}$$

(b) $\cos\frac{\theta}{2}$

Solution. Since $\frac{\theta}{2}$ is in Quadrant II, we know that $\cos\frac{\theta}{2} < 0$ so we will choose the negative (-) sign in the Half-Angle Formula:

$$\begin{aligned}
\cos\frac{\theta}{2} &= -\sqrt{\frac{1+\cos\theta}{2}} \\
&= -\sqrt{\frac{1+\left(-\frac{12}{13}\right)}{2}} \\
&= -\sqrt{\frac{1-\frac{12}{13}}{2}} \\
&= -\sqrt{\frac{\frac{13}{13}-\frac{12}{13}}{2}} \\
&= -\sqrt{\frac{\frac{1}{13}}{2}} \\
&= -\sqrt{\frac{1}{26}}.
\end{aligned}$$

(c) $\tan\frac{\theta}{2}$

Solution. Since $\frac{\theta}{2}$ is in Quadrant II, we know that $\tan\frac{\theta}{2} < 0$ so we will choose the negative (-) sign in the Half-Angle Formula:

$$\begin{aligned}
\tan\frac{\theta}{2} &= -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= -\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{1+\left(-\frac{12}{13}\right)}} \\
&= -\sqrt{\frac{1+\frac{12}{13}}{1-\frac{12}{13}}} \\
&= -\sqrt{\frac{\frac{13}{13}+\frac{12}{13}}{\frac{13}{13}-\frac{12}{13}}} \\
&= -\sqrt{\frac{\frac{25}{13}}{\frac{1}{13}}}
\end{aligned}$$

$$= -\sqrt{\frac{25}{13} \cdot \frac{13}{1}}$$
$$= -\sqrt{25}$$
$$= -5.$$

□

We can derive another formula for $\tan\frac{\theta}{2}$ that does not involve $\pm$.

Definition 3.3.10 Half-Angle Formulas for Tangent.

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

◊

Proof. We begin by first multiplying both sides of the Sine formula for Reducing Powers by 2 and halving the angle:

$$1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

and applying the double-angle formula to:

$$\sin\theta = \sin\left(2\cdot\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

By dividing the two preceding results we get

$$\frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

Thus

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}.$$

Similarly, it can be shown that

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}.$$

■

Example 3.3.11 Calculate $\tan\frac{\theta}{2}$ from Example 3.3.9 using the above formula.

Solution.

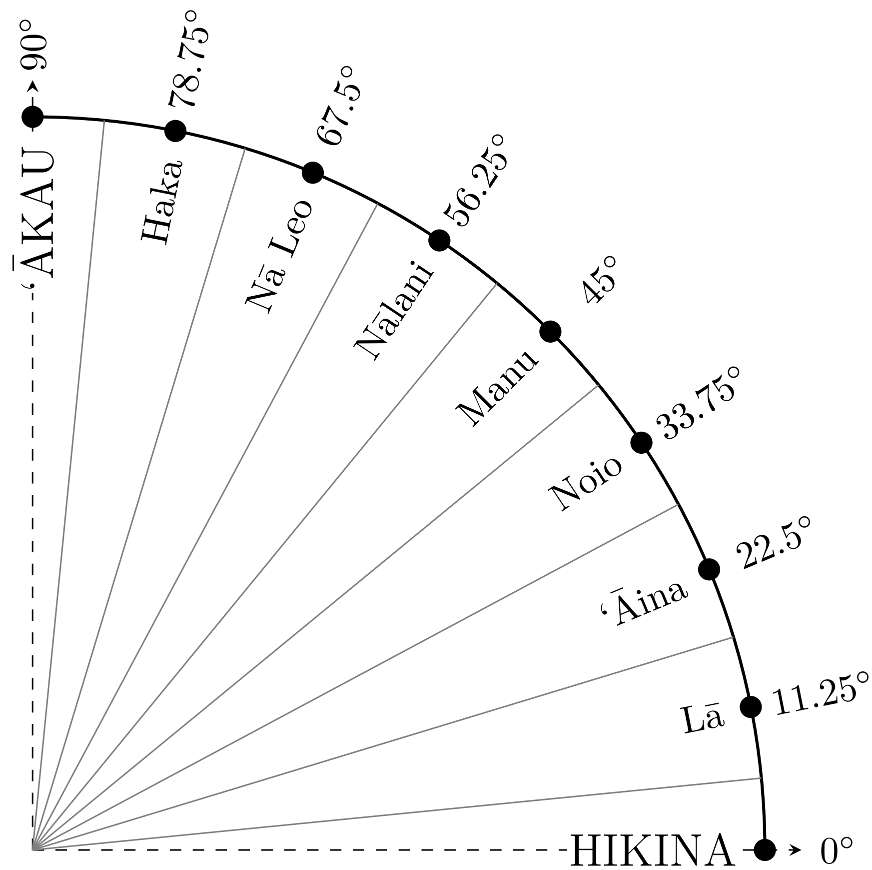
$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\left(-\frac{12}{13}\right)}{-\frac{5}{13}} = \frac{\frac{25}{13}}{-\frac{5}{13}} = \frac{25}{13}\cdot\left(-\frac{13}{5}\right) = -5.$$

□

Note: We obtained the same result for $\tan\frac{\theta}{2}$ as we did in Example 3.3.9. In this example, we did not have to determine whether $\tan\frac{\theta}{2}$ was positive or negative, however, we need to know the values of both $\sin\theta$ and $\cos\theta$.

3.3.4 Exercises

Exercise Group. The house ʻĀina is located at $22.5^\circ = \frac{45^\circ}{2}$ and the house Nā Leo is located at $67.5^\circ = \frac{135^\circ}{2}$. Use the half-angle formulas to evaluate the exact value of the given expression at each of these houses.



1. $\sin(22.5°)$

Answer. $\frac{1}{2}\sqrt{2-\sqrt{2}}$

2. $\tan(22.5°)$

Answer. $\sqrt{2}-1$

3. $\sin(67.5°)$

Answer. $\frac{1}{2}\sqrt{2+\sqrt{2}}$

4. $\cos(67.5°)$

Answer. $\frac{1}{2}\sqrt{2-\sqrt{2}}$

5. $\tan(67.5°)$

Answer. $\sqrt{2}+1$

6. Use the Addition Formula, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, to prove the double angle formula for cosine:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

7. Use the Pythagorean Identity ($\sin^2\theta+\cos^2\theta=1$) and the result from Exercise 3.3.4.6 to prove

$$\cos 2\theta = 1 - 2\sin^2\theta.$$

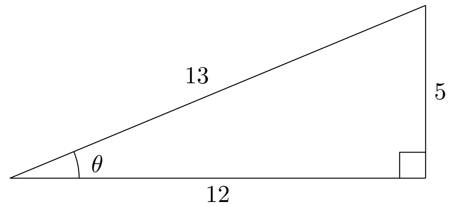
8. Use the Pythagorean Identity ($\sin^2\theta+\cos^2\theta=1$) and the result from Exercise 3.3.4.6 to prove

$$\cos 2\theta = 2\cos^2\theta - 1.$$

9. Use the addition formula, $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$, to prove the double angle formula for tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Exercise Group. Use the figure below to find the exact values for each of the following exercises.



10. $\sin(\theta)$

Answer. $\frac{5}{13}$

11. $\cos(\theta)$

Answer. $\frac{12}{13}$

12. $\tan(\theta)$

Answer. $\frac{5}{12}$

13. $\sin(2\theta)$

Answer. $\frac{120}{169}$

14. $\cos(2\theta)$

Answer. $\frac{119}{169}$

15. $\tan(2\theta)$

Answer. $\frac{120}{119}$

16. $\sin\left(\frac{\theta}{2}\right)$

Answer. $\sqrt{\frac{1}{26}}$

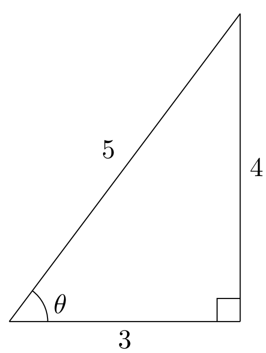
17. $\cos\left(\frac{\theta}{2}\right)$

Answer. $\sqrt{\frac{25}{26}}$

18. $\tan\left(\frac{\theta}{2}\right)$

Answer. $\frac{1}{5}$

Exercise Group. Use the figure below to find the exact values for each of the following exercises.



19. $\sin(\theta)$

Answer. $\frac{4}{5}$

20. $\cos(\theta)$

Answer. $\frac{3}{5}$

21. $\tan(\theta)$

Answer. $\frac{4}{3}$

22. $\sin(2\theta)$

Answer. $\frac{24}{25}$

23. $\cos(2\theta)$

Answer. $-\frac{7}{25}$

24. $\tan(2\theta)$

Answer. $-\frac{24}{7}$

25. $\sin\left(\frac{\theta}{2}\right)$

Answer. $\sqrt{\frac{1}{5}}$

26. $\cos\left(\frac{\theta}{2}\right)$

Answer. $\sqrt{\frac{4}{5}}$

27. $\tan\left(\frac{\theta}{2}\right)$

Answer. $\frac{1}{2}$

Exercise Group. Find the exact value of each expression given $\cos\theta = -\frac{3}{5}$ and θ is in Quadrant III.

28. $\sin(\theta)$

Answer. $-\frac{4}{5}$

29. $\tan(\theta)$

Answer. $\frac{4}{3}$

30. $\sin(2\theta)$

Answer. $\frac{24}{25}$

31. $\cos(2\theta)$

Answer. $-\frac{7}{25}$

32. $\tan(2\theta)$

Answer. $-\frac{24}{7}$

33. $\sin\left(\frac{\theta}{2}\right)$

Answer. $\sqrt{\frac{4}{5}}$

34. $\cos\left(\frac{\theta}{2}\right)$

Answer. $-\sqrt{\frac{1}{5}}$

35. $\tan\left(\frac{\theta}{2}\right)$

Answer. -2

Exercise Group. Find the exact value of each expression given $\sin\theta = -\frac{8}{17}$ and $270° < \theta < 360°$.

36. $\cos(\theta)$

Answer. $\frac{15}{17}$

37. $\tan(\theta)$

Answer. $-\frac{8}{15}$

38. $\sin(2\theta)$

Answer. $-\frac{240}{289}$

39. $\cos(2\theta)$

Answer. $\frac{161}{289}$

40. $\tan(2\theta)$

Answer. $-\frac{240}{161}$

41. $\sin\left(\frac{\theta}{2}\right)$

Answer. $\sqrt{\frac{1}{17}}$

42. $\cos\left(\frac{\theta}{2}\right)$

Answer. $-\sqrt{\frac{16}{17}}$

43. $\tan\left(\frac{\theta}{2}\right)$

Answer. $-\frac{1}{4}$

Exercise Group. Find the exact value of each expression given $\cos\theta = \frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$.

44. $\sin(\theta)$

Answer. $-\frac{\sqrt{5}}{3}$

45. $\tan(\theta)$

Answer. $-\frac{\sqrt{5}}{2}$

46. $\sin(2\theta)$

Answer. $-\frac{4\sqrt{5}}{9}$

47. $\cos(2\theta)$

Answer. $-\frac{1}{9}$

48. $\tan(2\theta)$

Answer. $4\sqrt{5}$

49. $\sin\frac{\theta}{2}$

Answer. $\sqrt{\frac{1}{6}}$

50. $\cos\frac{\theta}{2}$

Answer. $-\sqrt{\frac{5}{6}}$

51. $\tan\frac{\theta}{2}$

Answer. $-\sqrt{\frac{1}{5}}$

Exercise Group. Use the Half-Angle Formula to find the exact value of each of the following

52. $\tan 157.5°$

Answer. $1-\sqrt{2}$

53. $\sin 75°$

Answer. $\frac{1}{2}\sqrt{2+\sqrt{3}}$

54. $\tan 112.5°$

Answer. $-1-\sqrt{2}$

55. $\cos 15°$

Answer. $\frac{1}{2}\sqrt{2+\sqrt{3}}$

56. $\cos\frac{\pi}{8}$

Answer. $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \frac{1}{2}\sqrt{\sqrt{2}+2}$

57. $\tan\frac{11\pi}{12}$

Answer. $-2+\sqrt{3}$

58. $\sin \frac{7\pi}{12}$

Answer. $\frac{1}{2}\sqrt{2+\sqrt{3}}$

59. $\cos \frac{3\pi}{8}$

Answer. $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = \frac{1}{2}\sqrt{-\sqrt{2}+2}$

Exercise Group. Write each of the following as expressions that do not involve powers of sine or cosine greater than 1.

60. $\cos^4\theta$

Answer. $\frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$

61. $\sin^3(2\theta)$

Answer. $\frac{1}{2}\sin(2\theta) \cdot (1 - \cos(4\theta))$

62. $\sin^2\theta\cos^4\theta$

Answer. $\frac{1}{16}(1 + \cos(2\theta) - \cos(4\theta) - \cos(2\theta)\cos(4\theta))$

63. $\sin^4\theta\cos^2\theta$

Answer. $\frac{1}{16}(1 - \cos(2\theta) - \cos(4\theta) + \cos(2\theta)\cos(4\theta))$

64. Write $\sin^2\theta\cos^2\theta$ expressions that does not involve powers of sine or cosine greater than 1.

(a) Using the Reducing Powers Formula (Definition 3.3.5)

Answer. $\frac{1}{8}(1 - \cos(4\theta))$

(b) Using the Double Angle Formula (Definition 3.3.1) where $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

Answer. $\frac{1}{8}(1 - \cos(4\theta))$

65. Find the exact value of $\sin^2(15°)$

(a) Evaluate using the Reducing Powers Formula (Definition 3.3.5)

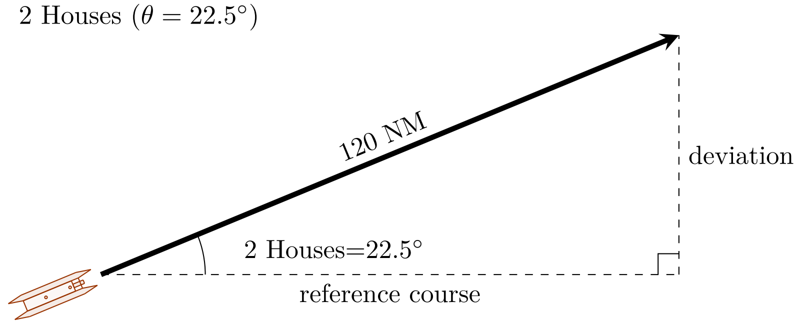
Answer. $\frac{2-\sqrt{3}}{4}$

(b) Evaluate using the Half-Angle Formula (Definition 3.3.7)

Answer. $\frac{2-\sqrt{3}}{4}$

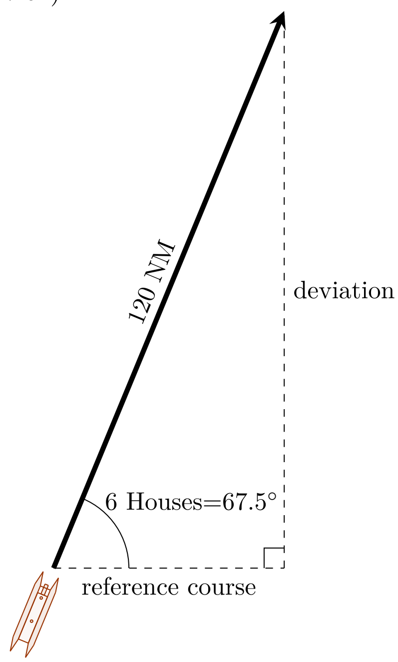
Exercise Group. Use half angle to find the exact deviation for the indicated angle, θ.

66. 2 Houses ($\theta = 22.5°$)



Answer. $60\sqrt{2-\sqrt{2}}$

67. 6 Houses ($\theta = 67.5°$)



Answer. $60\sqrt{2+\sqrt{2}}$

Exercise Group. Verify the identity

68. $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$

69. $\cos(2\theta) = \frac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta}$

70.
$(\sin^2\theta - 1)^2 = \cos(2\theta) + \sin^4\theta$

71. $\cos^2(3\theta) - \sin^2(3\theta) = \cos(6\theta)$

72. $\cos^4\theta - \sin^4\theta = \cos(2\theta)$

73. $\sin(6\theta) = 2\sin(3\theta)\cos(3\theta)$

74. $\cot(2\theta) = \frac{1-\tan^2\theta}{2\tan\theta}$

75. $\csc^2\left(\frac{\theta}{2}\right) = \frac{2}{1-\cos\theta}$

76. $\sec^2\left(\frac{\theta}{2}\right) = \frac{2}{1+\cos\theta}$

77. $\frac{2\tan\theta}{1+\tan^2\theta} = \sin(2\theta)$

3.4 Product-to-Sum and Sum-to-Product Formulas

In this section, we will learn how to convert sums of trigonometric functions to products of trigonometric functions, and vice versa. These techniques provide us with tools to simplify expressions and solve equations.

3.4.1 Product to Sum Formulas

Definition 3.4.1 Product to Sum Formulas.

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \\
\cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \\
\cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \\
\sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]
\end{aligned}$$

◊

Proof. We add the addition and subtraction formulas for cosine:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \\
+\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \\ \hline
\end{aligned}$$

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta.$$

Dividing both sides by 2, we get

$$\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]=\cos\alpha\cos\beta.$$

Next, we add the addition and subtraction formulas for sine:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
+\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \hline
\end{aligned}$$

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta.$$

Dividing both sides by 2, we get

$$\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]=\sin\alpha\cos\beta.$$

Next, we subtract the addition and subtraction formulas for sine:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
-\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \hline
\end{aligned}$$

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta.$$

Dividing both sides by 2, we get

$$\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]=\cos\alpha\sin\beta.$$

Finally, we subtract the addition and subtraction formulas for cosine:

$$\begin{array}{r} \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ -\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \hline \end{array}$$

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta.$$

Dividing both sides by 2, we get

$$\frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] = \sin\alpha\sin\beta.$$

■

Example 3.4.2 Express the product of $\cos(2x)\cos(5x)$ as a sum or difference of sine and cosine with no products.

Solution. Using the formula we get

$$\begin{aligned} \cos(2x)\cos(5x) &= \frac{1}{2}[\cos(2x+5x) + \cos(2x-5x)] \\ &= \frac{1}{2}[\cos(7x) + \cos(-3x)]. \end{aligned}$$

This satisfies the requirement of expressing the product of $\cos(2x)\cos(5x)$ as a sum or difference of sine and cosine with no products. However, we can simplify it further.

Since cosine is an even function, $\cos(-3x) = \cos(3x)$. Thus, we can simplify the expression to:

$$\cos(2x)\cos(5x) = \frac{1}{2}[\cos(7x) + \cos(3x)].$$

□

Example 3.4.3 Express the product of $\sin(6\theta)\cos(4\theta)$ as a sum or difference of sine and cosine with no products.

Solution. Using the formula we get

$$\begin{aligned} \sin(6\theta)\cos(4\theta) &= \frac{1}{2}[\sin(6\theta+4\theta) + \sin(6\theta-4\theta)] \\ &= \frac{1}{2}[\sin(10\theta) + \sin(2\theta)]. \end{aligned}$$

□

Remark 3.4.4 Negative Angles. In the previous example, both forms $\cos(2x)\cos(5x) = \frac{1}{2}[\cos(7x) + \cos(-3x)]$ and $\cos(2x)\cos(5x) = \frac{1}{2}[\cos(7x) + \cos(3x)]$ are valid representations of the answer. However, it is more common to write the form where the angles are positive because it simplifies the expression and aligns with standard conventions for representing trigonometric identities. Positive angles are often preferred for clarity and consistency in mathematical notation. Negative angles can be transformed into positive angles using the even-odd properties of trigonometric functions (Definition 1.5.22).

3.4.2 Sum to Product Formulas

Definition 3.4.5 Sum-to-Product Formula.

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

◊

Proof. We first let $\alpha = \frac{u+v}{2}$ and $\beta = \frac{u-v}{2}$. Then

$$\alpha+\beta = \frac{u+v}{2} + \frac{u-v}{2} = \frac{2u}{2} = u$$

and

$$\alpha-\beta = \frac{u+v}{2} - \frac{u-v}{2} = \frac{2v}{2} = v.$$

Substituting these values for α, β, $\alpha+\beta$, and $\alpha-\beta$ into the Product-to-Sum Formulas, we get

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin(u)+\sin(v)]$$
$$\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin(u)-\sin(v)]$$
$$\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\cos(u)+\cos(v)]$$
$$\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) = \frac{1}{2}[\cos(v)-\cos(u)].$$

Multiplying both sides by 2 and substituting u with α and v with β, we arrive at the Sum-to-Product Formulas, where we negate the last equation. ■

Example 3.4.6 Express the sum $\sin(8x)+\sin(2x)$ as a product of sines or cosines.

Solution. Using the formula we get

$$\begin{aligned}\sin(8x)+\sin(2x) &= 2\sin\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)\\ &= 2\sin\left(\frac{10x}{2}\right)\cos\left(\frac{6x}{2}\right)\\ &= 2\sin(5x)\cos(3x).\end{aligned}$$

□

Example 3.4.7 Express the difference $\cos(3t)-\cos(5t)$ as a product of sines or cosines.

Solution. Using the formula we get

$$\cos(3t)-\cos(5t) = -2\sin\left(\frac{3t+5t}{2}\right)\sin\left(\frac{3t-5t}{2}\right)$$

$$= -2\sin\left(\frac{8t}{2}\right)\sin\left(\frac{-2t}{2}\right)$$
$$= -2\sin(4t)\sin(-t)$$
$$= 2\sin(4t)\sin(t).$$

□

3.4.3 Exercises

Exercise Group. Express each product as a sum or difference of sine and cosine.

1. $\sin(3x)\cos(5x)$

Answer. $\frac{1}{2}[\sin(8x) + \sin(-2x)] = \frac{1}{2}[\sin(8x) - \sin(2x)]$

2. $\sin(7t)\cos(-2t)$

Answer. $\frac{1}{2}[\sin(5t) + \sin(9t)]$

3. $\cos(-3t)\sin(7t)$

Answer. $\frac{1}{2}[\sin(4t) - \sin(-10t)] = \frac{1}{2}[\sin(4t) + \sin(10t)]$

4. $\cos(9\theta)\sin(6\theta)$

Answer. $\frac{1}{2}[\sin(15\theta) - \sin(3\theta)]$

5. $\cos(-4x)\cos(6x)$

Answer. $\frac{1}{2}[\cos(2x) + \cos(-10x)] = \frac{1}{2}[\cos(2x) + \cos(10x)]$

6. $\cos(2\theta)\cos(4\theta)$

Answer. $\frac{1}{2}[\cos(6\theta) + \cos(-2\theta)] = \frac{1}{2}[\cos(6\theta) + \cos(2\theta)]$

7. $\sin(-\theta)\sin(8\theta)$

Answer. $\frac{1}{2}[\cos(-9\theta) - \cos(7\theta)] = \frac{1}{2}[\cos(9\theta) - \cos(7\theta)]$

8. $\sin(6t)\sin(-3t)$

Answer. $\frac{1}{2}[\cos(9t) - \cos(3t)]$

Exercise Group. Express each sum or difference as a product.

9. $\sin(-2\theta) + \sin(-9\theta)$

Answer. $2\sin\left(-\frac{11\theta}{2}\right)\cos\left(\frac{7\theta}{2}\right) = -2\sin\left(\frac{11\theta}{2}\right)\cos\left(\frac{7\theta}{2}\right)$

10. $\sin(5x) + \sin(7x)$

Answer. $2\sin(6x)\cos(-x) = 2\sin(6x)\cos(x)$

11. $\sin(4\theta) - \sin(-7\theta)$

Answer. $2\cos\left(-\frac{3\theta}{2}\right)\sin\left(\frac{11\theta}{2}\right) = 2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{11\theta}{2}\right)$

12. $\sin(3x) - \sin(-4x)$

Answer. $2\cos\left(-\frac{x}{2}\right)\sin\left(\frac{7x}{2}\right) = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{7x}{2}\right)$

13. $\cos(8\theta) + \cos(-5\theta)$

Answer. $2\cos\left(\frac{3\theta}{2}\right)\cos\left(\frac{13\theta}{2}\right)$

14. $\cos(9t) + \cos(2t)$

Answer. $2\cos\left(\frac{11t}{2}\right)\cos\left(\frac{7t}{2}\right)$

15. $\cos(6\theta) - \cos(8\theta)$

Answer. $-2\sin(7\theta)\sin(-\theta) = 2\sin(7\theta)\sin(\theta)$

16. $\cos(6t) - \cos(-3t)$

Answer. $-2\sin\left(\frac{9t}{2}\right)\sin\left(\frac{3t}{2}\right)$

Exercise Group. Find the exact value of each expression.

17. $\sin(195°)\cos(105°)$

Answer. $\frac{1}{2}\left(-\frac{\sqrt{3}}{2} + 1\right)$

18. $\cos(225°)\cos(195°)$

Answer. $\frac{\sqrt{3}}{4} + \frac{1}{4}$

19. $\sin(195°) - \sin(75°)$

Answer. $-\frac{\sqrt{6}}{2}$

20. $\cos(165°) - \cos(105°)$

Answer. $-\frac{\sqrt{2}}{2}$

21. $\sin(285°) + \sin(195°)$

Answer. $-\frac{\sqrt{6}}{2}$

22. $\cos(255°) + \cos(15°)$

Answer. $\frac{\sqrt{2}}{2}$

Exercise Group. Verify the identity

23. $\sin\theta + \sin(3\theta) = 4\sin\theta\cos^2\theta$

24. $\cos(3\theta) + \cos\theta = 2\left(\cos^3\theta - \sin^2\theta\cos\theta\right)$

25. $6\cos(5\theta)\sin(6\theta) = 3\sin(11\theta) + 3\sin(\theta)$

26. $\frac{\sin\theta+\sin(3\theta)}{2\sin(2\theta)} = \cos\theta$

27. $\frac{\cos\theta+\cos(3\theta)}{2\cos(2\theta)} = \cos\theta$

28. $\frac{\cos\theta-\cos(3\theta)}{\sin\theta+\sin(3\theta)} = \tan\theta$

3.5 Basic Trigonometric Equations

Trigonometric equations are equations involving trigonometric functions such as sine, cosine, and tangent. These equations seek to find specific values, known as **solutions**, that satisfy the equation.

Trigonometric functions are inherently periodic, meaning they repeat their values at regular intervals. As a result, trigonometric equations may have multiple solutions due to this periodic nature. In fact, some equations may have infinitely many solutions.

To address all possible solutions, we use a technique known as a **general solution**. This method involves initially identifying solutions within a single period of the trigonometric function. We then extend these solutions by adding integer multiples of the period of the trigonometric function.

In this section, we will explore various techniques for effectively solving trigonometric equations, including methods for finding general solutions.

3.5.1 Solving Equations with a Single Trigonometric Function

Example 3.5.1 Solve the equation

$$\sin\theta = \frac{1}{2}.$$

Solution. To solve the equation $\sin\theta = \frac{1}{2}$, our initial instinct might lead us to take the inverse sine of both sides:

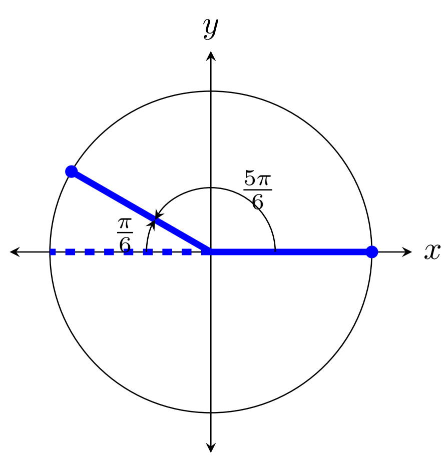
$$\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{1}{2}\right)$$

resulting in

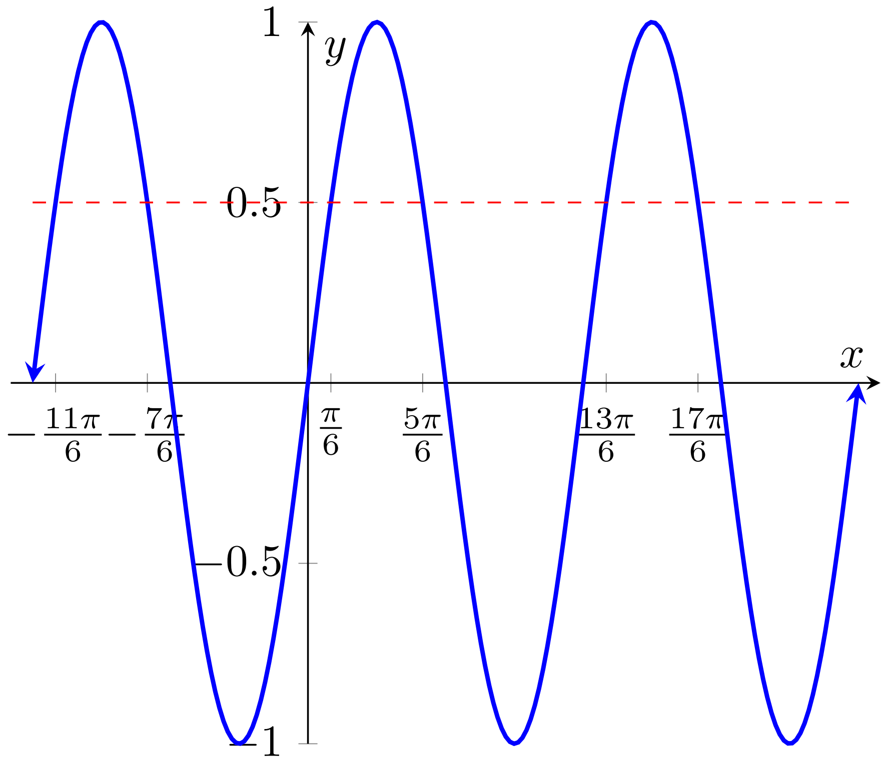
$$\theta = \frac{\pi}{6}.$$

While this is a valid solution, it's important to recognize that there are additional solutions to consider.

Since the sine function is positive in both Quadrant I and Quadrant II, we can find another solution in Quadrant II by using the reference angle $\frac{\pi}{6}$ and the methods described in Subsection 1.5.2. Therefore, the equivalent angle in Quadrant II is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.



However, these two angles, $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$, are not the only solutions. Recall that the sine function has a period of 2π, meaning that adding or subtracting any integer multiples of 2π to these angles will also give us solutions. For example, $\theta = \frac{\pi}{6} + 4\pi$ and $\theta = \frac{5\pi}{6} - 10\pi$ are both solutions.



Thus, the general solution to $\sin\theta = \frac{1}{2}$ can be expressed as:

$$\theta = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{6} + 2k\pi,$$

where k is any integer. $\square$

Example 3.5.2 Solve the equation $2\cos\theta + \sqrt{2} = 0$. List six solutions.

Solution. First we will isolate $\cos\theta$.

$$\begin{aligned} 2\cos\theta + \sqrt{2} &= 0 \\ 2\cos\theta &= -\sqrt{2} \\ \cos\theta &= -\frac{\sqrt{2}}{2}. \end{aligned}$$

Thus we have the general solution $\theta = \frac{3\pi}{4} + 2k\pi$ or $\theta = \frac{5\pi}{4} + 2k\pi$ for any integer k.

To get specific solutions, we select specific values of k:

$$\begin{aligned} k = -1: \theta &= \frac{3\pi}{4} + 2(-1)\pi = -\frac{5\pi}{4}, & \theta &= \frac{5\pi}{4} + 2(-1)\pi = -\frac{3\pi}{4} \\ k = 0: \theta &= \frac{3\pi}{4} + 2(0)\pi = \frac{3\pi}{4}, & \theta &= \frac{5\pi}{4} + 2(0)\pi = \frac{5\pi}{4} \\ k = 1: \theta &= \frac{3\pi}{4} + 2(1)\pi = \frac{11\pi}{4}, & \theta &= \frac{5\pi}{4} + 2(1)\pi = \frac{13\pi}{4} \end{aligned}$$

$\square$

3.5.2 Solving Trigonometric Equations with Square Terms

Example 3.5.3 Solve $\cos^2\theta = \frac{1}{2}$ on the interval $0 \leq \theta < 2\pi$.

Solution. We will first solve for $\cos\theta$. We begin by taking the square root of both sides of the equation and simplify:

$$\sqrt{\cos^2\theta} = \pm\sqrt{\frac{1}{2}}$$
$$\cos\theta \pm \frac{1}{\sqrt{2}}$$

From Table 1.5.18, when $\cos\theta = \frac{1}{\sqrt{2}}$, we have $\theta = \frac{\pi}{4}$ or $\theta = \frac{7\pi}{4}$; and when $\cos\theta = -\frac{1}{\sqrt{2}}$, we have $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$.

Thus, our solutions are:

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

□

3.5.3 Solving Trigonometric Equations by Factoring

Example 3.5.4 Solve $\cos^2 x - 4\cos x + 3 = 0$.

Solution. To solve this equation, let's make a substitution to simplify it. We'll let $y = \cos x$, so the equation becomes $y^2 - 4y + 3 = 0$.

Factoring the quadratic equation, we obtain $(y-3)(y-1) = 0$. Setting each factor equal to zero, we find two potential solutions: $y = 3$ or $y = 1$. We are not done because we need to solve for x and not y.

Substituting back $\cos x$ for y, we find that $\cos x = 3$ is not a valid solution, as the range of cosine is limited to [-1, 1]. However, $\cos x = 1$ yields a solution of $x = 0$ for one period.

Therefore, the general solution to the equation is:

$$x = 0 + 2k\pi = 2k\pi,$$

where k is any integer. □

Example 3.5.5 Solve $2\cos\theta\sin\theta + \sqrt{3}\cos\theta = 0$.

Solution. We begin by factoring $\cos\theta$:

$$2\cos\theta\sin\theta + \sqrt{3}\cos\theta = 0$$
$$\cos\theta(2\sin\theta + \sqrt{3}) = 0.$$

Thus we get two equations: $\cos\theta = 0$ and $2\sin\theta + \sqrt{3} = 0$.

From $\cos\theta = 0$ we get $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$.

From the second equation, we isolate $\sin\theta$:

$$2\sin\theta + \sqrt{3} = 0$$
$$2\sin\theta = -\sqrt{3}$$
$$\sin\theta = -\frac{\sqrt{3}}{2}.$$

Thus we get $\theta = \frac{4\pi}{3}$ or $\theta = \frac{5\pi}{3}$.

We get the general solutions by adding integer multiples of 2π to get

$$\theta = \frac{\pi}{2} + 2k\pi, \quad \theta = \frac{3\pi}{2} + 2k\pi, \quad \theta = \frac{4\pi}{3} + 2k\pi, \quad \theta = \frac{5\pi}{3} + 2k\pi$$

where k is any integer. □

3.5.4 Solving a Trigonometric Equation with a Calculator

Example 3.5.6 Use a calculator to solve $3\tan\theta = 2$ on the interval $0 \le \theta < 2\pi$. Express the answer in radians, rounded to two decimals.

Solution. We begin by isolating $\tan\theta$:

$$\tan\theta = \frac{2}{3}$$

Next, we take the inverse tangent and use a calculator to obtain

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.588002603548.$$

Rounding to two decimals, we get $\theta = 0.59$ radians, which is in Quadrant I since $0 < 0.59 < \frac{\pi}{2}$. Another angle where $\tan\theta = \frac{2}{3}$ is in Quadrant III. Using the methods in Subsection 1.5.2 we get the other angle: $\theta = 0.59 + \pi$. □

3.5.5 Exercises

Exercise Group. Solve each equation on the interval $0 \le \theta < 2\pi$

1. $\sin\theta = \frac{\sqrt{3}}{2}$
Answer. $\frac{\pi}{3}$, $\frac{2\pi}{3}$

2. $\tan\theta = -1$
Answer. $\frac{3\pi}{4}$, $\frac{7\pi}{4}$

3. $\cot\theta = -\sqrt{3}$
Answer. $\frac{5\pi}{6}$, $\frac{11\pi}{6}$

4. $\csc\theta = 2$
Answer. $\frac{\pi}{6}$, $\frac{5\pi}{6}$

5. $\sec\theta = -2$
Answer. $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

6. $\cos\theta = \frac{1}{\sqrt{2}}$
Answer. $\frac{\pi}{4}$, $\frac{7\pi}{4}$

7. $\csc\theta + \sqrt{2} = 0$
Answer. $\frac{5\pi}{4}$, $\frac{7\pi}{4}$

8. $6\cos\theta + 1 = -2$
Answer. $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

9. $2\tan\theta + 2\sqrt{3} = 0$
Answer. $\frac{2\pi}{3}$, $\frac{5\pi}{3}$

10. $2\cot\theta + 8 = 6$
Answer. $\frac{3\pi}{4}$, $\frac{7\pi}{4}$

11. $2\sin\theta + 1 = 0$
Answer. $\frac{7\pi}{6}$, $\frac{11\pi}{6}$

12. $\sec\theta - \sqrt{2} = 0$
Answer. $\frac{\pi}{4}$, $\frac{7\pi}{4}$

Exercise Group. Solve each equation, giving the general formula for each solution. List six specific solutions.

13. $\sin\theta = -\frac{1}{\sqrt{2}}$
Answer. $\frac{5\pi}{4} + 2k\pi$, $\frac{7\pi}{4} + 2k\pi$; $-\frac{3\pi}{4}$, $-\frac{\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, $\frac{13\pi}{4}$, $\frac{15\pi}{4}$

14. $\cos\theta = \frac{1}{2}$
Answer. $\frac{\pi}{3} + 2k\pi$, $\frac{5\pi}{3} + 2k\pi$; $-\frac{5\pi}{3}$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, $\frac{11\pi}{3}$

15. $\tan\theta = -1$
Answer. $\frac{3\pi}{4} + k\pi$; $-\frac{5\pi}{4}$, $-\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, $\frac{11\pi}{4}$, $\frac{15\pi}{4}$

16. $\cot\theta = -\sqrt{3}$
Answer. $\frac{5\pi}{6} + k\pi$; $-\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{11\pi}{6}$, $\frac{17\pi}{6}$, $\frac{23\pi}{6}$, $\frac{29\pi}{6}$

Exercise Group. Solve each equation on the interval $0 \le \theta < 2\pi$. Express your answer in radians, rounded to two decimals.

17. $\sin\theta = -0.82$

Answer. 4.10, 5.32

18. $\cos\theta = 0.3$

Answer. 1.27, 5.02

19. $\tan\theta = 2.5$

Answer. 1.19, 4.33

20. $\cot\theta = -5$

Answer. 2.94, 6.09

Exercise Group. Solve each equation on the interval $0 \le \theta < 2\pi$.

21. $4\cos^2\theta - 3 = 0$

Answer. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

22. $3\csc^2\theta - 4 = 0$

Answer. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

23. $\tan^2\theta - 1 = 0$

Answer. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

24. $2\cot^2\theta - 6 = 0$

Answer. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

25. $\sec^2\theta - 4 = 0$

Answer. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

26. $2\cos^2\theta - 1 = 0$

Answer. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

27. $(2\cos\theta - 1)(\csc\theta + 2) = 0$

Answer. $\frac{\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

28. $(\tan\theta - \sqrt{3})(\cot\theta + 1) = 0$

Answer. $\frac{\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}$

29. $4\sin^2\theta - 2\sin\theta = 0$

Answer. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

30. $2\cos^2\theta - \sqrt{3}\cos\theta = 0$

Answer. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6}$

31. $\sin^3\theta - \sin\theta = 0$

Solution. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

32. $2\cos^2\theta + \cos\theta - 1 = 0$

Answer. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

33. $2\sin^2\theta - 7\sin\theta + 3 = 0$

Answer. $\frac{\pi}{6}, \frac{5\pi}{6}$

34. $3\tan^3\theta - \tan\theta = 0$

Answer. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$

35. $\cos^2\theta + 2\cos\theta + 1 = 0$

Answer. π

36. $\csc^5\theta - 4\csc\theta = 0$

Answer. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

37. $2\sin^2\theta - 3\sin\theta - 2 = 0$

Answer. $\frac{7\pi}{6}, \frac{11\pi}{6}$

38. $2\cos^2\theta - 3\cos\theta + 1 = 0$

Answer. $0, \frac{\pi}{3}, \frac{5\pi}{3}$

3.6 Trigonometric Equations - Advanced Techniques

In this section, we will solve trigonometric equations using various trigonometric identities, equations involving multiple angles, and graphical methods.

3.6.1 Solving a Trigonometric Equation by Using Fundamental Identities

Example 3.6.1 Solve $2\sin^2 x + 3\cos x - 3 = 0$.

Solution. To solve the equation $2\sin^2 x + 3\cos x - 3 = 0$, we want to first write it in terms of only cosine or only sine. By the Pythagorean Identity, we substitute $\sin^2 x = 1 - \cos^2 x$, resulting in:

$$\begin{aligned}
2\sin^2 x + 3\cos x - 3 &= 0\\
2(1-\cos^2 x) + 3\cos x - 3 &= 0\\
2 - 2\cos^2 x + 3\cos x - 3 &= 0\\
-2\cos^2 x + 3\cos x - 1 &= 0\\
2\cos^2 x - 3\cos x + 1 &= 0\\
(2\cos x - 1)(\cos x - 1) &= 0.
\end{aligned}$$

Thus $\cos x = \frac{1}{2}$ or $\cos x = 1$.

Solving for x in the first equation gives $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$, while the second equations gives $x = 0$ for one period. Finding all solutions, we arrive at $x = \frac{\pi}{3} + 2k\pi$, $\frac{5\pi}{3} + 2k\pi$, or $x = 0 + 2k\pi = 2k\pi$ for any integer k. □

Remark 3.6.2 If factoring trigonometric functions is difficult, we can try substituting a simpler term and then factor. For example, consider the expression $2\cos^2 x - 3\cos x + 1$ in the previous example. If factoring this expression directly is not clear, we can use the substitution $A = \cos x$. This transforms the expression into $2A^2 - 3A + 1$, which may be easier to factor. Once factored, we can substitute $\cos x$ back in for A to obtain the final solution.

Example 3.6.3 Solve the equation $\sin(2\theta) + \cos\theta = 0$ on the interval $0 \le \theta < 2\pi$.

Solution. Notice the first term has 2θ so we will begin by using the double-angle formula:

$$\begin{aligned}
\sin(2\theta) + \cos\theta &= 0\\
2\sin\theta\cos\theta + \cos\theta &= 0.
\end{aligned}$$

Then we factor $\cos\theta$ from the terms:

$$\cos\theta(2\sin\theta + 1) = 0.$$

Thus we get

$$\cos\theta = 0 \quad \text{or} \quad 2\sin\theta + 1 = 0.$$

When $\cos\theta = 0$ we get $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.
When $2\sin\theta + 1 = 0$, $\sin\theta = -\frac{1}{2}$, thus $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.
Combining these, the solutions are

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

□

3.6.2 Solving a Trigonometric Equation with Multiples of an Angle

Example 3.6.4 Solve the equation $2\cos(2\theta) - \sqrt{3} = 0$ on the interval $0 \le \theta < 2\pi$.

Solution. First we will isolate $\cos(2\theta)$. We begin by rearranging the equation:

$$\begin{aligned} 2\cos(2\theta) - \sqrt{3} &= 0 \\ 2\cos(2\theta) &= \sqrt{3} \\ \cos(2\theta) &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Since $\cos(2\theta)$ equals $\frac{\sqrt{3}}{2}$ for angles $\frac{\pi}{6}$ and $\frac{11\pi}{6}$, the general solutions for 2θ are

$$2\theta = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad 2\theta = \frac{11\pi}{6} + 2k\pi$$

for any integer k.

Note that we have only solved for 2θ. Dividing both sides by 2 to find θ, we obtain:

$$\theta = \frac{\pi}{12} + k\pi \quad \text{or} \quad \theta = \frac{11\pi}{12} + k\pi.$$

The restriction $0 \le \theta < 2\pi$ gives us the following solutions:

$$\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}.$$

□

Remark 3.6.5 In this example, our angle was a double-angle: 2θ. When dealing with multiple-angle trigonometric functions, such as $\cos(2\theta)$, it's essential to understand their graphical behavior. According to Definition 2.1.28, the graph of $\cos(2\theta)$ undergoes a horizontal compression by a factor of 2, and its period is $\frac{2\pi}{2} = \pi$. Since we are asked to find solutions on the interval $0 \le \theta < 2\pi$, we will need to consider two periods. Thus, we have four solutions (2 solutions for each period):

$$\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}.$$

In general, if the trigonometric function has an angle $k\theta$, for some number k, we will need to consider the effect of this multiple angle on the period and number of solutions, ensuring that we adjust our solutions accordingly to cover all possible solutions within the given interval.

Example 3.6.6 Solve the equation $3\tan\frac{\theta}{2} - \sqrt{3} = 0$ on the interval $0 \le \theta < 2\pi$.

Solution. We begin by isolating $\tan\frac{\theta}{2}$:

$$\begin{aligned} 3\tan\frac{\theta}{2} - \sqrt{3} &= 0 \\ 3\tan\frac{\theta}{2} &= \sqrt{3} \\ \tan\frac{\theta}{2} &= \frac{\sqrt{3}}{3}. \end{aligned}$$

Since $\tan\frac{\theta}{2}$ equals $\frac{\sqrt{3}}{3}$ for angles $\frac{\pi}{6}$ and $\frac{7\pi}{6}$, the general solutions for $\frac{\theta}{2}$ are

$$\frac{\theta}{2} = \frac{\pi}{6} + k\pi \quad \text{and} \quad \frac{\theta}{2} = \frac{7\pi}{6} + k\pi$$

for any integer k.

Solving for θ, we multiply both sides by 2 to obtain:

$$\theta = \frac{\pi}{3} + 2k\pi \quad \text{and} \quad \theta = \frac{7\pi}{3} + 2k\pi.$$

Considering our restriction $0 \leq \theta < 2\pi$, we have only one solution:

$$\theta = \frac{\pi}{3}.$$

□

Remark 3.6.7 In this example, the angle was $\frac{\theta}{2}$, which stretches the graph of the tangent function. When we have an angle of the form $\frac{\theta}{2}$, it stretches the period of the tangent function by a factor of 2. Thus, for the given interval, we have only half a period to consider instead of the full period.

3.6.3 Solving a Trigonometric Equation with a Graphing Utility

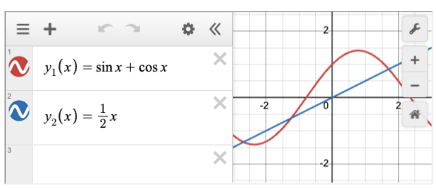
Sometimes we will encounter equations where an exact solution is not possible. However, we may be able to get an approximation to the solution by graphing the equation.

Example 3.6.8 Use a graphing utility to find the solutions to the equation $\sin x + \cos x = \frac{1}{2}x$. Express the answers in radians, rounded to two decimals.

Solution. To find the solution to $\sin x + \cos x = \frac{1}{2}x$, we graph the left-hand side and the right-hand side of the equation and identify their intersections. Let y_1 represent the curve for the left-hand side and y_2 represent the curve for the right-hand side:

$$y_1 = \sin x + \cos x, \quad y_2 = \frac{1}{2}x.$$

Use a graphing utility to plot y_1 and y_2.



Standalone

Figure 3.6.9 Plotting $y_1 = \sin x + \cos x$ and $y_2 = \frac{1}{2}x$, corresponding to the left-hand and right-hand sides of the equation, respectively. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to explore the interactive version online.

Next, we may need to zoom in or out to better visualize the behavior of the curves. To find their intersection points, calculators often have a TRACE or INTERSECT button or command. In Desmos Graphing Calculator[1] we can click on either curve, and the points of intersection will be highlighted. Hovering the cursor over the intersection will display the coordinates of that point.

The equation $\sin x + \cos x = \frac{1}{2}x$ has three solutions, which correspond to the points of intersection between the curves $y_1 = \sin x + \cos x$ and $y_2 = \frac{1}{2}x$. The x-values of these intersections are

$$x = -2.68, -1.24, 1.71.$$

□

3.6.4 Exercises

Exercise Group. Solve each equation on the interval $0 \leq \theta < 2\pi$.

1. $\sin^2\theta - \cos^2\theta = 0$

Answer. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

2. $\cos^2\theta - \sin^2\theta = 1 + \sin\theta$

Answer. $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$

3. $3\cot^2\theta - 4\csc\theta = 1$

Answer. $\frac{\pi}{6}, \frac{5\pi}{6}$

4. $\sin^2\theta = 5\cos\theta + 5$

Answer. π

5. $\cos^2\theta = 3 - 3\sin\theta$

Answer. $\frac{\pi}{2}$

6. $2\sin^2\theta = 3\cos\theta + 3$

Answer. $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$

Trigonometric Equations Involving Multiples of an Angle. Solve the given trigonometric equation on the interval $0 \leq \theta < 2\pi$.

7. $\cot 2\theta = -\sqrt{3}$

Answer. $\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

8. $\sin 4\theta = \frac{\sqrt{3}}{2}$

Answer. $\frac{\pi}{12}, \frac{\pi}{6}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{13\pi}{12}, \frac{7\pi}{6}, \frac{19\pi}{12}, \frac{5\pi}{3}$

9. $\sqrt{2}\cos 2\theta = 1$

Answer. $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

10. $\csc 3\theta = 2$

Answer. $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

11. $\sec\frac{3\theta}{2} = -\sqrt{2}$

Answer. $\frac{\pi}{2}, \frac{5\pi}{6}, \frac{11\pi}{6}$

12. $\cos\frac{\theta}{2} - 1 = 0$

Answer. 0

Trigonometric Equations Involving Addition or Subtraction Formula. Use the Addition and Subtraction Formulas to solve each equation on the interval $0 \leq \theta < 2\pi$.

13. $\sin\theta\cos 2\theta + \cos\theta\sin 2\theta = \frac{\sqrt{3}}{2}$

Answer. $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$

14. $\sin 3\theta\cos 2\theta - \cos 3\theta\sin 2\theta = -\frac{1}{2}$

Answer. $\frac{7\pi}{6}, \frac{11\pi}{6}$

15. $\cos 3\theta\cos\theta + \sin 3\theta\sin\theta = -\frac{\sqrt{2}}{2}$

Answer. $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

16. $\cos\theta\cos 3\theta - \sin\theta\sin 3\theta = 1$

Answer. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Trigonometric Equations Involving Double-Angle or Half-Angle Formula. Use the Double-Angle and Half-Angle Formulas to solve each equation on the interval $0 \leq \theta < 2\pi$.

17. $\sin 2\theta = \cos\theta$

Answer. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

18. $\cos 2\theta = \cos\theta$

Answer. $0, \frac{2\pi}{3}, \frac{4\pi}{3}$

19. $\cos 2\theta + \cos\theta - 2 = 0$

Answer. 0

20. $\tan 2\theta = -2\sin\theta$

Answer. $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

21. $\cos 2\theta + 2 = 2\sin^2\theta$

Answer. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

22. $\tan\frac{\theta}{2} = \sin\theta$

Answer. $0, \frac{\pi}{2}, \frac{3\pi}{2}$

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[1] `DesmosGraphingCalculator`

Trigonometric Equations Involving Sum-to-Product Formula. Use the Sum-to-Product Formulas to solve each equation on the interval $0 \leq \theta < 2\pi$.

23. $\sin 3\theta + \sin\theta = 0$

Answer. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

24. $\sin 6\theta - \sin 2\theta = \cos 4\theta$

Answer. $\frac{\pi}{12}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{12}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{12}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{17\pi}{12}, \frac{13\pi}{8}, \frac{15\pi}{8}$

25. $\cos 4\theta - \cos 2\theta = 0$

Solution. $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

26. $\cos 3\theta + \cos\theta = 0$

Solution. $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

Exercise Group. Use a graphing utility to solve each equation. Express your solutions in radians, rounded to two decimals.

27. $\sin(2x) = 4\cos x + x$

Answer. $x = -1.34, 2.63, 3.90$

28. $\sin x - x = \cos x$

Answer. $x = -1.26$

29. $x^2 = \cos x$

Answer. $x = -0.82, 0.82$

30. $x^3 + 2x^2 = \cos(2x)$

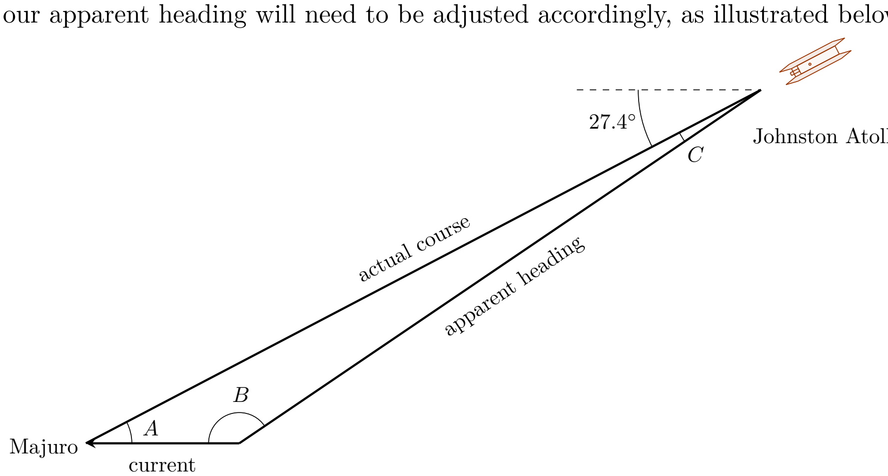
Answer. $x = -2.11, -0.56, 0.48$

Chapter 4

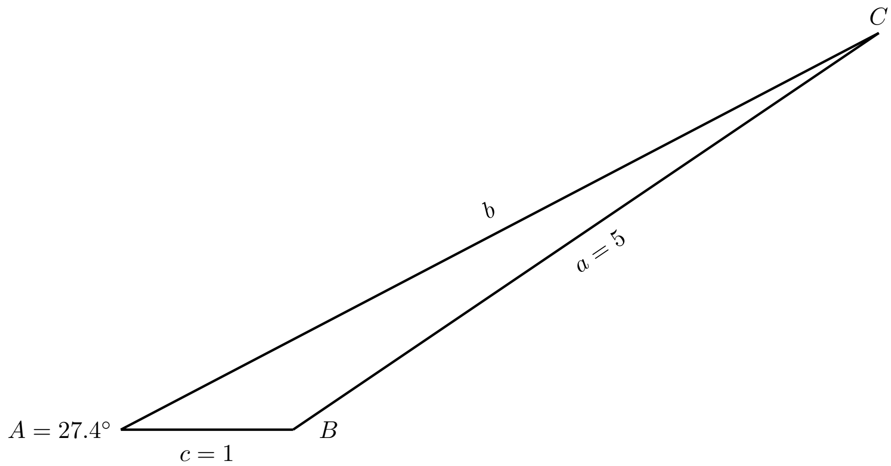
Applications of Trigonometry

4.1 The Law of Sines

Recall Example 2.4.18 in Section 2.4 where Alingano Maisu is sailing to her new home in Satawal. For the first leg of the voyage, we calculated that to sail from Johnston Atoll to Majuro, we would need to head towards House ʻĀina Kona (27.4°). However, in reality, currents can push the waʻa off course. Let's consider a scenario where a current of 1 knot is flowing in the direction of the house Komohana (West). If we were to face Majuro and sail straight, the current would push us to the west of our destination. To ensure we arrive at Majuro, our apparent heading will need to be adjusted accordingly, as illustrated below.



Now, suppose we sail at 5 knots. What heading should we aim for to compensate for a 1-knot current? We note that by alternate interior angles, $A = 27.4°$. Let's represent this situation with a triangle where the sides represent speed of the waʻa and the current, allowing us to disregard the distance traveled.

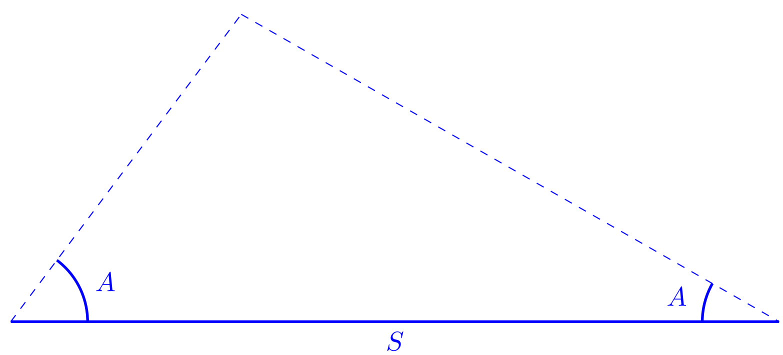


It's important to note that this triangle is not a right triangle; instead, it's classified as an **oblique triangle**, meaning it does not have any right angles. Triangles can be further categorized as either *obtuse* or *acute*. An **obtuse triangle** contains one angle greater than 90°, while an **acute triangle** consists of all angles less than 90°. Solving such triangles requires different methods than those used for solving the right triangle in previous chapters. In this chapter, we'll learn techniques for solving oblique triangles, starting with the Law of Sines and the Law of Cosines.

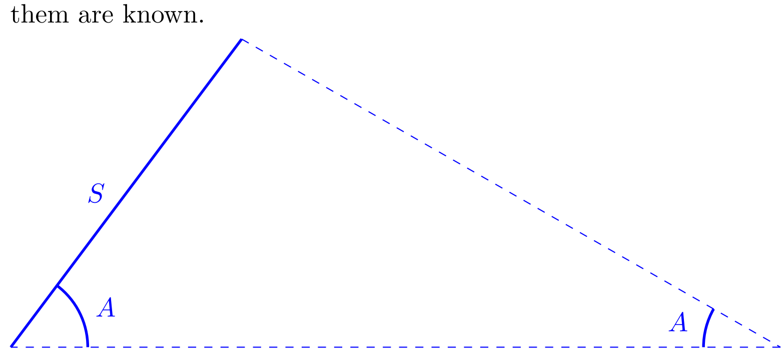
To solve an oblique triangle, we need to be given three pieces of information, which can take the following forms:

1. One side and two angles.

   - *Angle-Side-Angle (ASA)*: Two angles and the side between them are known.

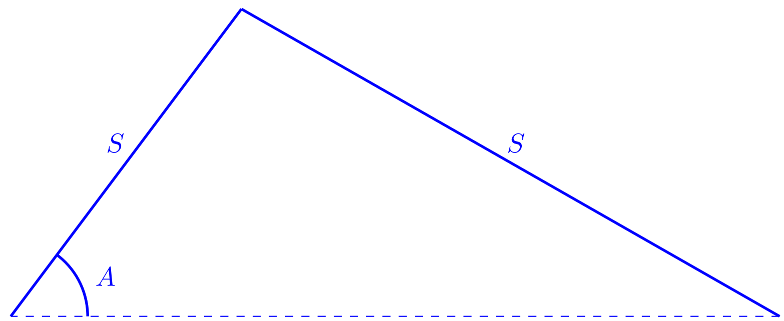


   - *Side-Angle-Angle (SAA)*: Two angles and a side that is not between them are known.

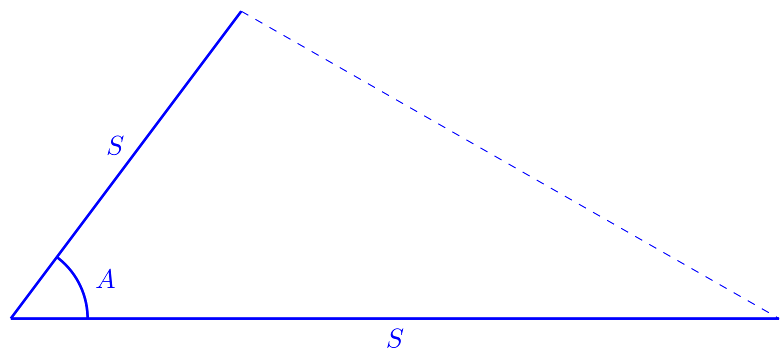


2. Two sides and one angle.

    - *Side-Side-Angle (SSA)*: Two sides and the angle that is not between them are known.

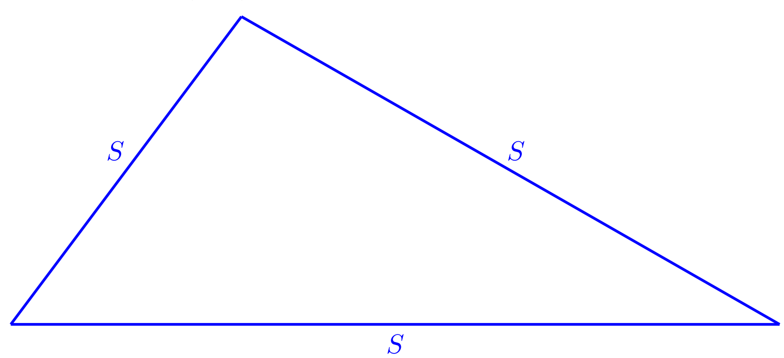


    - *Side-Angle-Side (SAS)*: Two sides and the angle between them are known.



3. Three sides.

    - *Side-Side-Side (SSS)*: All three sides of the triangle are known.



In this section, we explore techniques for solving triangles when provided with one side and two angles (ASA or SAA) and when given two sides along with the angle not between them (SSA). Triangles falling under the categories of SAS and SSS will be discussed in more detail in the Section 4.2.

4.1.1 Law of Sines

The **Law of Sines** states that for any triangle, the ratios of the lengths of the sides to the sine of their corresponding opposite angles are equal. When provided with certain information about the sides and angles of a triangle, this fundamental principle becomes a valuable tool for determining the remaining sides and angles.

Theorem 4.1.1 Law of Sines. *In triangle ABC, we have*

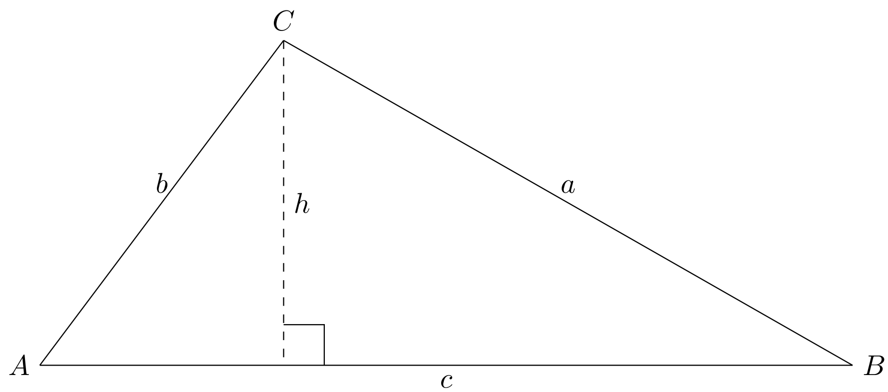
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

which may also be written in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The first form is more convenient when we are trying to find an angle and the second form is more convenient when we are trying to find a side.

Proof. In any triangle, we can draw an **altitude**, a line through a vertex and perpendicular to the side opposite the vertex.



Using the right triangles, we have $\sin A = \frac{h}{b}$ and $\sin B = \frac{h}{a}$. Solving both equations for h, we get $h = b \sin A$ and $h = a \sin B$. Since these equations both equal h, we can set them equal to each other:

$$b \sin A = a \sin B.$$

Dividing both sides by ab, we get

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

Using the same method, we can get:

$$\frac{\sin A}{a} = \frac{\sin C}{c} \text{ and } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Putting these together, we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

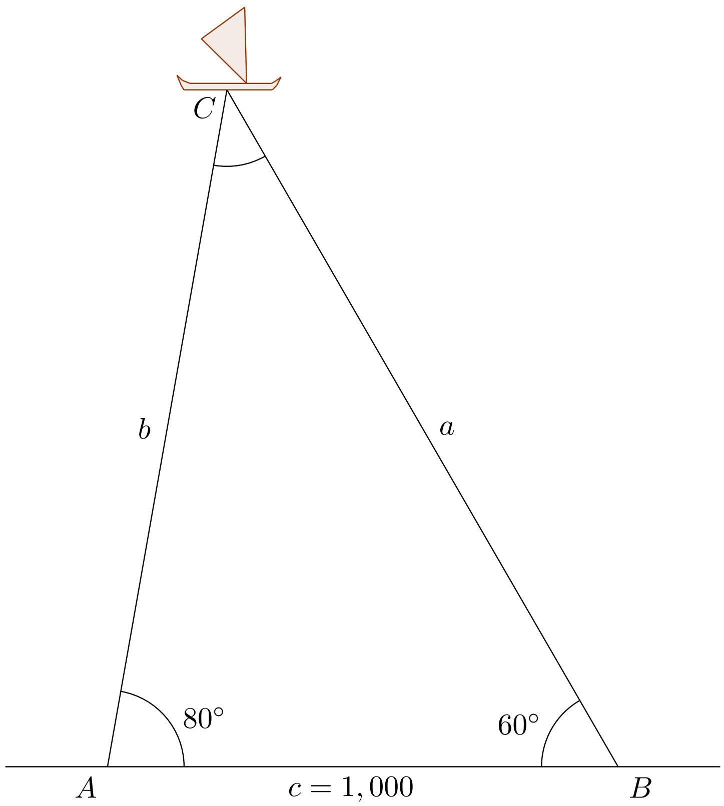
■

Remark 4.1.2 When solving triangles, it is often useful to use the fact that the sum of the angles in a triangle equals 180°. That is,

$$A + B + C = 180°.$$

4.1.2 Solving ASA Triangles

Example 4.1.3 Using Law of Sines (ASA). The waʻa Kānehūnāmoku is returning from a morning sail, training students in the ways of sailing. Two observers standing 1,000 feet apart on the shore at Kāneʻohe Bay spot Kānehūnāmoku. Observer A determines that the angle between the waʻa and Observer B is 80° while Observer B determines that the angle between the waʻa and Observer A is 60°. How far is Kānehūnāmoku from each observer and what is the angle formed by the two observers (round the answer to the nearest foot)?



Solution. In this triangle, we know the values of two angles and the side between them, so we have an Angle-Side-Angle (ASA) triangle. We know that $A+B+C=180°$. So $80°+60°+C=140°+C=180°$ so $C=40°$. To solve for a, we will use the Law of Sines. Either form for the Law of Sines can be used to solve this problem, but since we are solving for a side, it would be easier and require fewer steps if we use the second form for the Law of Sines.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{c}{\sin C}\\ \frac{a}{\sin 80°} &= \frac{1,000}{\sin 40°}\end{aligned}$$

Multiplying both sides by $\sin 80°$,

$$a = \frac{1,000 \sin 80°}{\sin 40°} \approx 1,532 \text{ ft.}$$

To solve for b, we have two options to use for the Law of Sines:

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad \text{or} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Because our answer for a was rounded, using the first option may lead to rounding errors so it is best to use the second option because it has known values that we were originally given in the problem ($c = 1,000$). Thus we will use

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

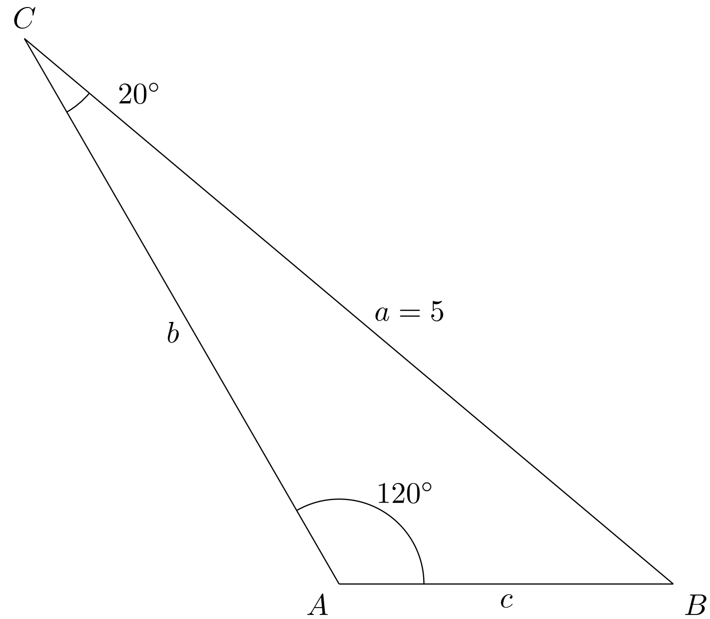
So we get

$$b = \frac{c \sin B}{\sin C} = \frac{1,000 \sin 60^\circ}{\sin 40^\circ} \approx 1,347 \text{ ft}.$$

Thus Kānehūnāmoku is 1,347 feet from Observer A and 1,532 feet from Observer B, and the angle between the two observers is 40°. □

4.1.3 Solving SAA Triangles

Example 4.1.4 Using Law of Sines (SAA). For the triangle shown below with $A = 120°$, $C = 20°$, and $a = 5$, find the remaining sides. Round the answer to two decimal places.



Solution. In this triangle, we know the values of two angles and the side between them, so we have a Side-Angle-Angle (SAA) triangle. The third angle can be calculated by subtracting the known angles from 180°:

$$B = 180° - A - C = 180° - 120° - 20° = 40°.$$

By using the first form of the Law of Sines

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad \text{and} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

we have

$$b = \frac{a \sin B}{\sin A} = \frac{5 \sin 40°}{\sin 120°} \approx 3.71$$

and

$$c = \frac{a \sin C}{\sin A} = \frac{5 \sin 20°}{\sin 120°} \approx 1.97.$$

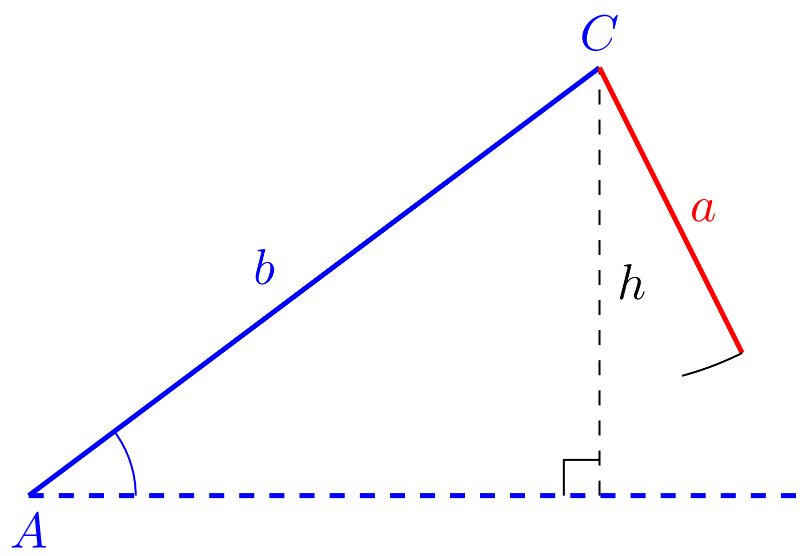
□

4.1.4 Solving SSA Triangles: The Ambiguous Case

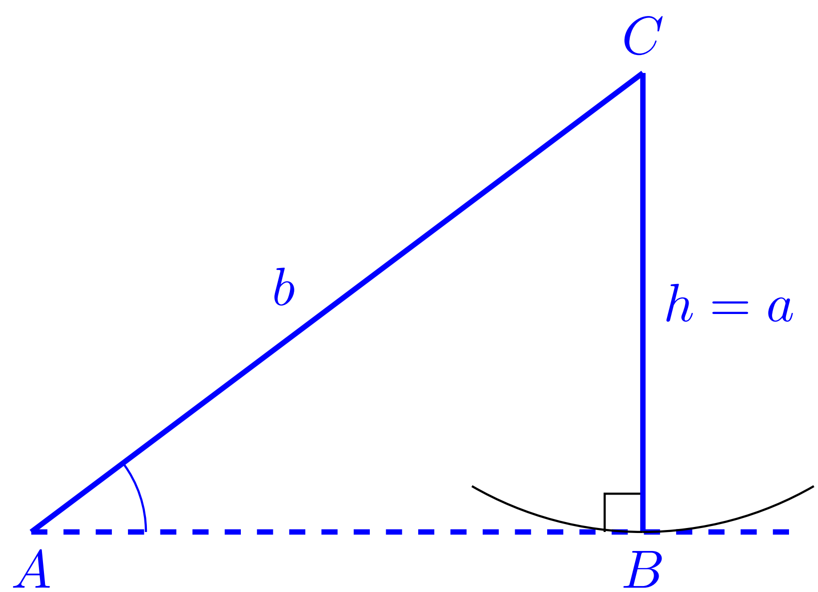
Although the Law of Sines can be used to solve oblique triangles, there may be cases that would give us one triangle, two triangles, or even no solution. Such cases are known as the *ambiguous case*. This occurs when we know two sides and an angle that is opposite to one of the given sides.

Definition 4.1.5 The Ambiguous Case. Consider a triangle where a, b, and A are given. The altitude is $h = b \sin A$. We have the following possibilities:

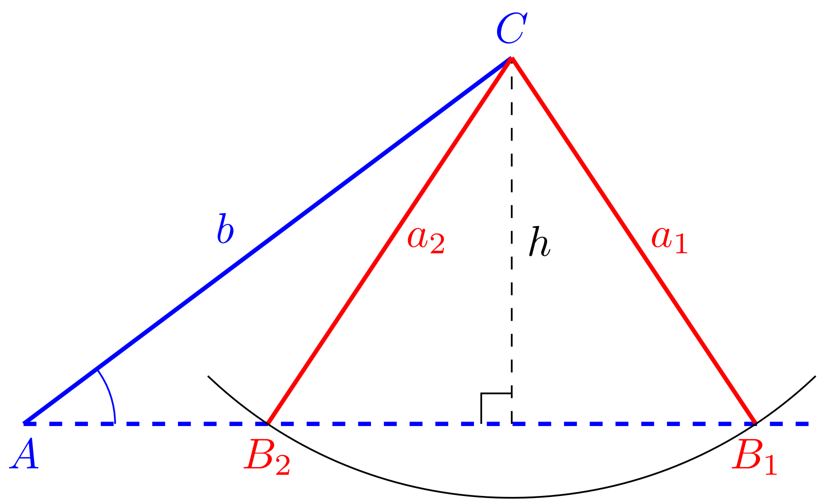
1. A is acute and $a < h = b \sin A$: *No triangle*

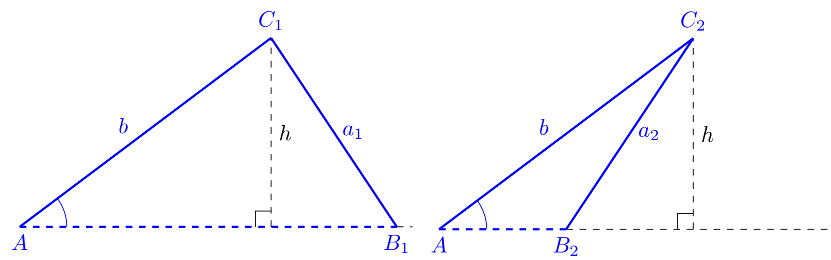


2. A is acute and $a = h = b\sin A$: *One right triangle*

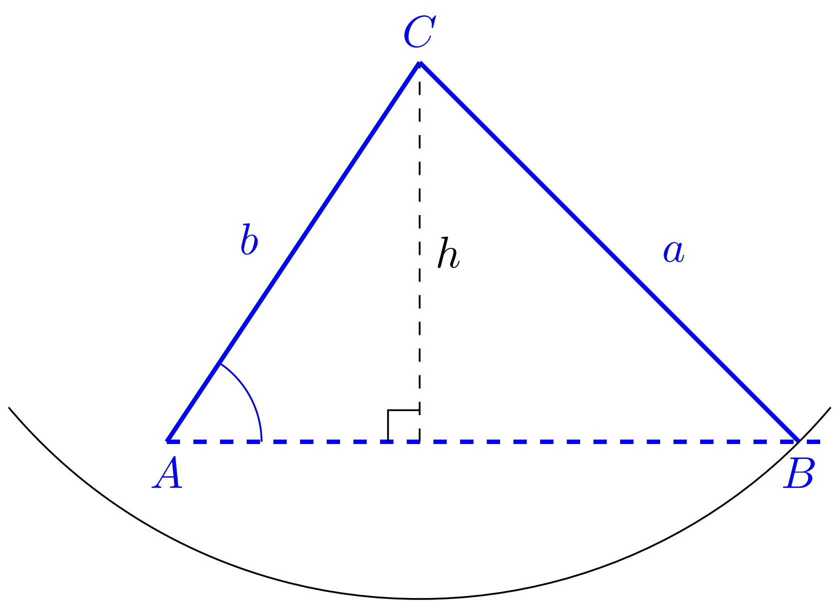


3. A is acute and $h = b\sin A < a < b$: *Two triangles* ($\triangle AB_1C_1$ and $\triangle AB_2C_2$)

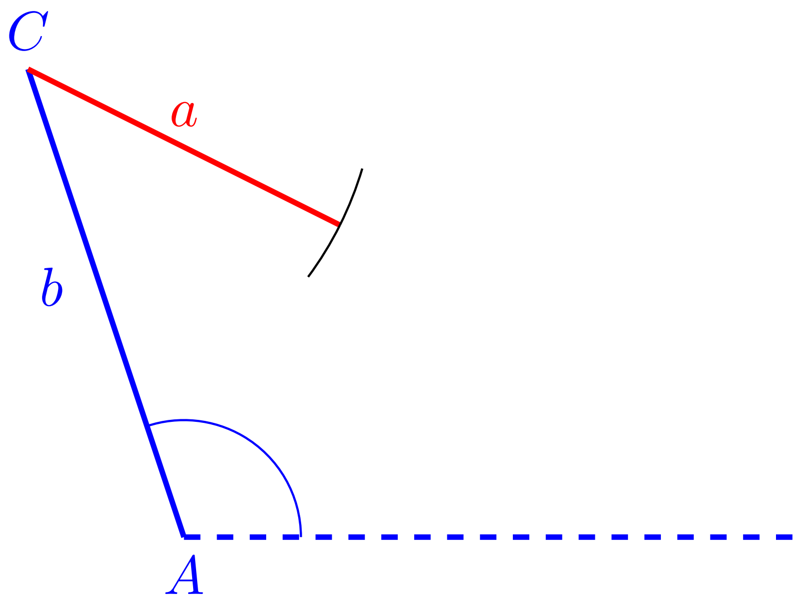




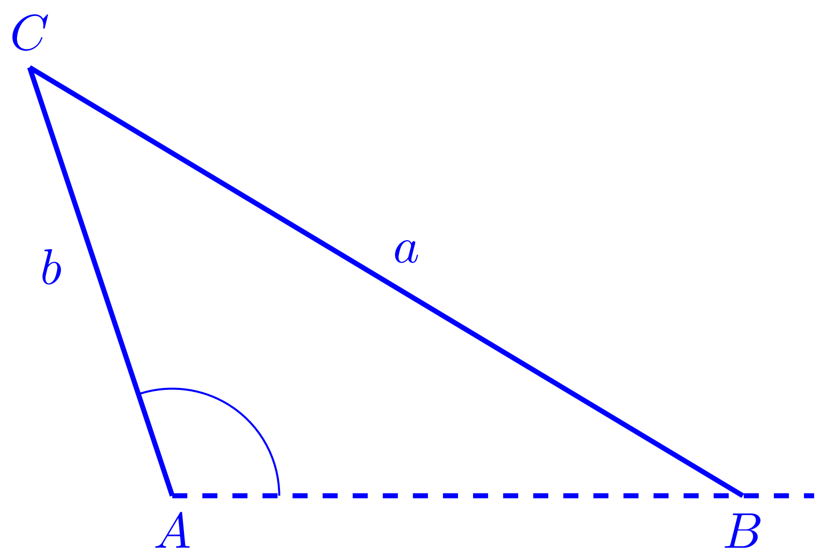
4. A is acute and $a \geq b$: *One triangle*



5. A is obtuse and $a \leq b$: *No triangle*



6. A is obtuse and $a > b$: *One triangle*

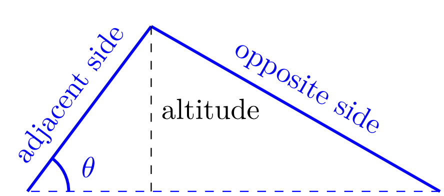


◊

Remark 4.1.6 Generalizing the Ambiguous Case. Not every time we encounter a SSA triangle will we be given the angle A and sides a and b. Sometimes we may be given angle B and sides a and b. In these cases, we cannot use the equations and inequalities listed above to identify how many triangles we will get.

To generalize the ambiguous case of SSA, we are given a triangle with:

1. a known angle, θ;
2. a side **adjacent** to θ;
3. a side **opposite** to θ.

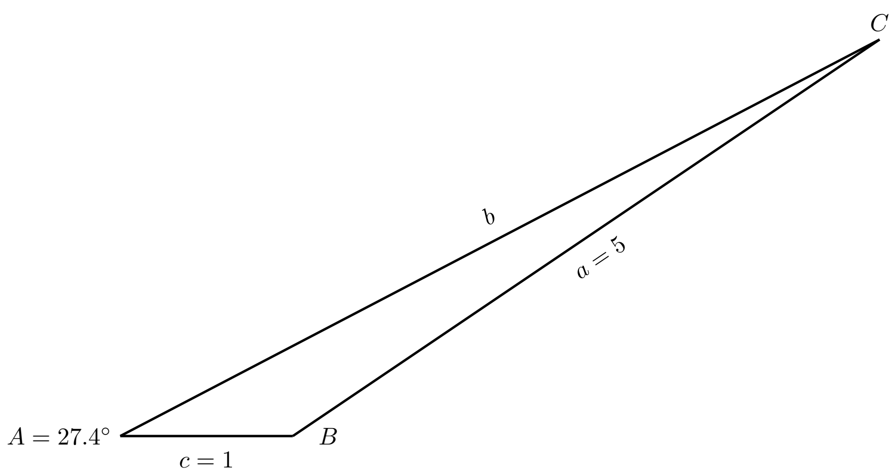


If we calculate the **altitude** $h = (\text{adjacent side}) \cdot \sin\theta$, then Table 4.1.7 summarizes the possible cases.

Table 4.1.7 Possibilities for the ambiguous case (SSA)

Case	θ	Condition on Opposite Side	Number of Triangles
1	acute	opposite side < altitude	None
2	acute	opposite side = altitude	One (right triangle)
3	acute	altitude < opposite side < adjacent side	Two
4	acute	opposite side ≥ adjacent side	One
5	obtuse	opposite side ≤ adjacent side	None
6	obtuse	opposite side > adjacent side	One

Example 4.1.8 Using Law of Sines (SSA) one-solution. We can now return to the example at the start of the section where the Alingano Maisu must account for a current and we had the following triangle:



Which house do we need to sail into to account for the current?

Solution. Referring to Table 4.1.7, our known angle ($A = 27.4°$) is acute, the side opposite of our angle ($a = 5$) is greater than the side adjacent to the angle ($c = 1$) so we have Case 4 and know that we have one triangle.

Using the first form of the Law of Sines

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

we get

$$\sin C = c \cdot \frac{\sin A}{a} \approx 1 \cdot \frac{\sin 27.4°}{5} \approx 0.092.$$

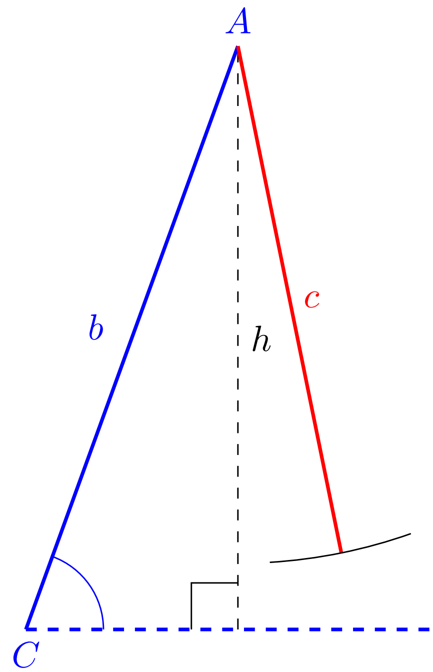
Taking the inverse sine we get

$$C = \sin^{-1} 0.092 \approx 5.3°.$$

Now that we know the value of C, we will need to add that to the heading that we determined in the last section $(207.4°)$ to get $5.3° + 207.4° = 212.7°$. Next we refer to the Star Compass with angles (Figure 1.2.4) to conclude we will need to sail towards the House Noio Kona. □

Example 4.1.9 Using Law of Sines (SSA) no solution. Solve the triangle if $C = 70°$, $b = 6$, and $c = 5$.

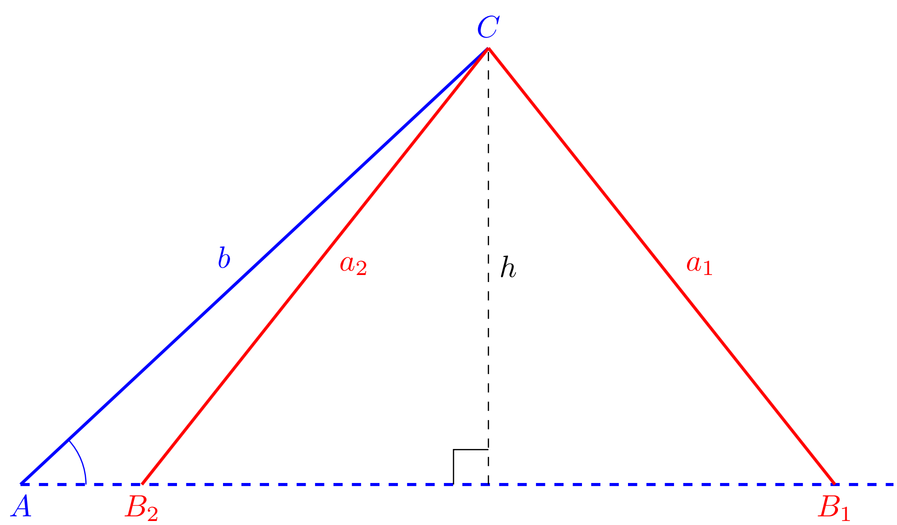
Solution. The altitude of the triangle is $h = b \sin C = 6 \sin 70° \approx 5.64$. Since C is an acute angle and the altitude ($h \approx 5.64$) is greater than the side opposite of the angle ($c = 5$), we have Case 1 of Table 4.1.7, thus there is no triangle.



□

Example 4.1.10 Using Law of Sines (SSA) two solutions. Solve the triangle if $a = 6$, $b = 7$, and $A = 40°$.

Solution. The altitude of the triangle is $h = b \sin A = 7 \sin 40° \approx 4.50$. Since A is an acute angle and the altitude ($h \approx 4.50$) is less than the opposite side ($a = 6$) which is less than the adjacent side ($b = 7$), this satisfies Case 3 of Table 4.1.7, thus there are two triangles.



Using the first form for the Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

we get

$$\sin B = \frac{b \sin A}{a} = \frac{7 \sin 40^\circ}{6} \approx 0.75.$$

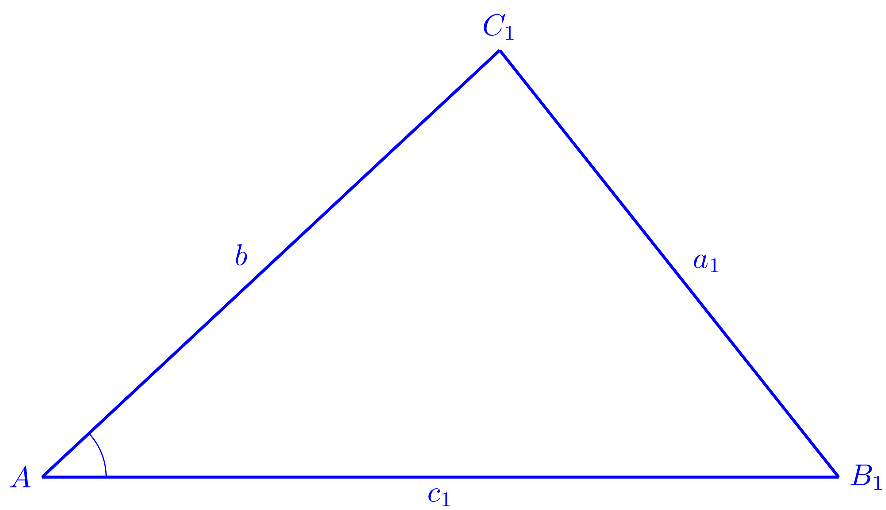
Here we have that $\sin B$ is positive and we know that sine is positive in Quadrant I and II so our value for B is between 0° and 180°. Taking the inverse sine we get our first angle

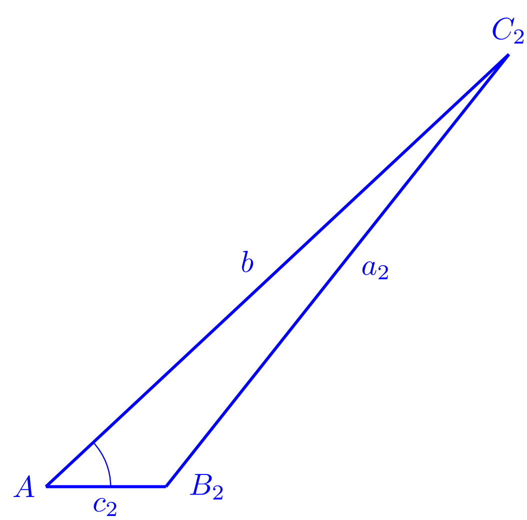
$$B_1 = \sin^{-1}(0.75) \approx 48.58^\circ,$$

which is in Quadrant I. To calculate the angle in Quadrant II, we subtract the reference angle from 180° to get

$$B_2 = 180^\circ - B_1 = 180^\circ - 48.58^\circ = 131.42^\circ.$$

So we now have two triangles: AB_1C_1 and AB_2C_2.





To solve triangle AB_1C_1, be begin by finding C_1:

$$C_1 \approx 180^\circ - A - B_1 = 180^\circ - 40^\circ - 48.58^\circ = 91.42^\circ.$$

To find the side c_1, we use the second form for the Law of Sines

$$\frac{c_1}{\sin C_1} = \frac{a}{\sin A}.$$

Thus

$$c_1 = \frac{a \sin C_1}{\sin A} \approx \frac{6 \sin 48.58°}{\sin 40°} \approx 9.33.$$

To solve triangle AB_2C_2, we begin by finding C_2

$$C_2 \approx 180° - A - B_2 = 180° - 40° - 131.42° = 8.58°.$$

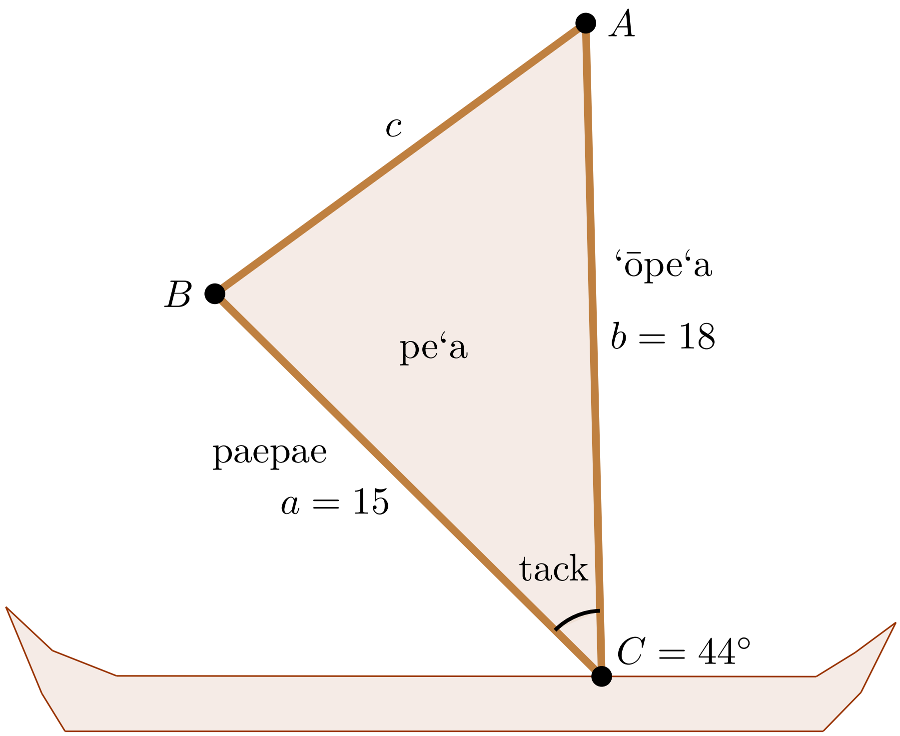
To find the side c_2, we use the second form for the Law of Sines to get

$$c_2 = \frac{a \sin C_2}{\sin A} \approx \frac{6 \sin 8.58°}{\sin 40°} \approx 1.39.$$

□

4.1.5 Area of an oblique triangle

After heavy use, we determine it's time to replace the peʻa (sail) of our waʻa. Before we buy the new peʻa, we need to determine its area. We know the length of the ʻōpeʻa (spar) is $b = 18$ feet, the length of the paepae (boom) is $a = 15$ feet, and the tack or the angle between them is $C = 44°$. What is the area of the peʻa?



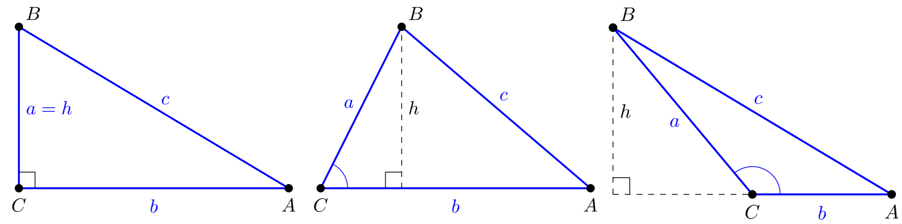
At this point, we don't have a formula for the area of an oblique triangle. However, we can apply the sine function to an oblique triangle to develop a new formula for the area of a triangle.

Definition 4.1.11 Area of a Triangle. Given two sides of a triangle and their included angle, the area of the triangle is given by:

$$\begin{aligned} \text{Area} &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} ac \sin B \end{aligned}$$

$$= \frac{1}{2} bc \sin A.$$

In other words, the area of a triangle equals one half of the product of two sides and the sine of their included angle.



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Proof. Recall that the area of any triangle is Area $= \frac{1}{2}bh$ where b is the base and h is the height of a triangle. In a right triangle, the height is the length of one side. However, in an oblique triangle, the height is not immediately known. If the angle C is acute, then using the property that

$$\sin C = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{a},$$

the height is given by

$$h = a \sin C.$$

If the angle C is obtuse, we use the reference angle for C, $180° - C$ to get

$$h = a \sin(180° - C).$$

By the difference formula we get $\sin(180° - C) = \sin C$ and thus

$$h = a \sin C.$$

In a right triangle, $C = 90°$ and since $\sin 90° = 1$, we can also write

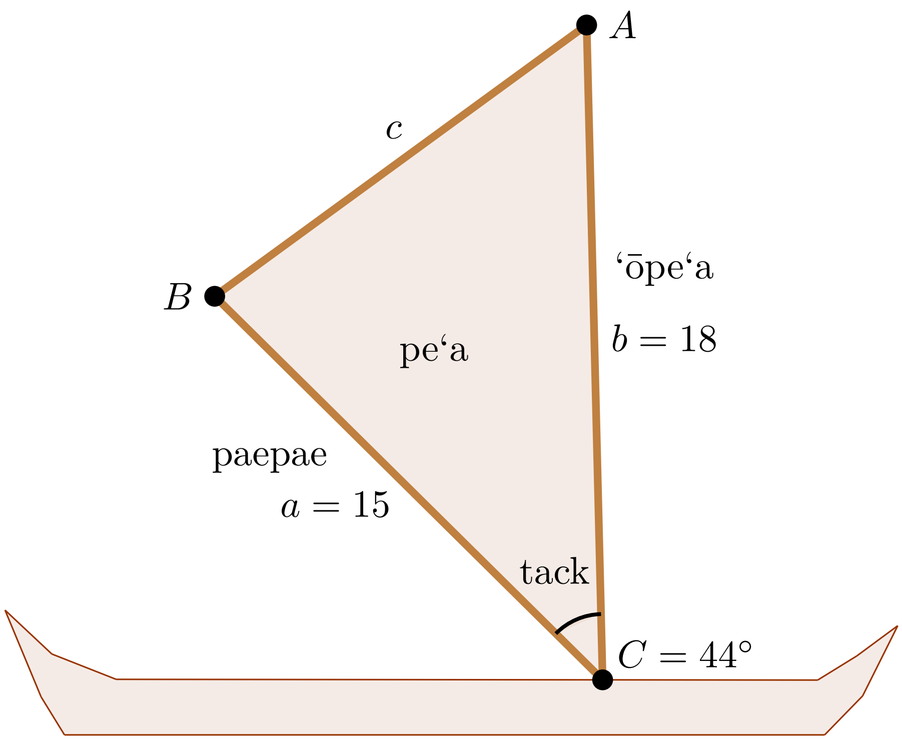
$$h = a \sin C = a \sin 90° = a \cdot 1 = a.$$

Thus regardless of the triangle, we get

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} b(a \sin C) = \frac{1}{2} ab \sin C.$$

The other formulas are obtained using a similar method. ■

Example 4.1.12 Area of a sail. We can now calculate the area of the sail.



Solution. Using the formula, the area is

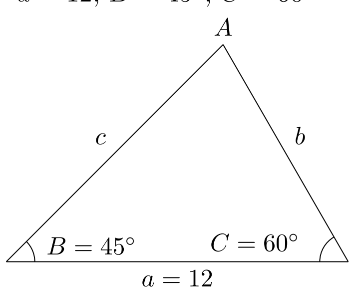
$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}(15)(18)\sin 44° \approx 93.8\text{ft}^2$$

Thus we will need to order a sail that is 93.8ft^2. □

4.1.6 Exercises

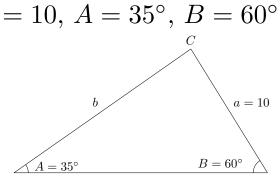
Exercise Group. Use the Law of Sines to solve each triangle. Express angles in degrees rounded to one decimal and sides rounded to two decimals.

1. $a = 12$, $B = 45°$, $C = 60°$



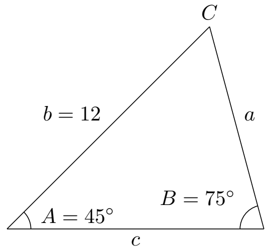
Answer. $b \approx 8.78$, $c \approx 10.76$, $A = 75°$

2. $a = 10$, $A = 35°$, $B = 60°$



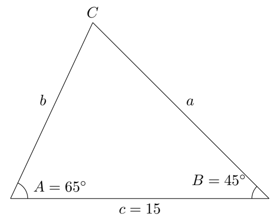
Answer. $b \approx 15.10$, $c \approx 17.37$, $C = 85°$

3. $b = 12,\ A = 45°,\ B = 75°$



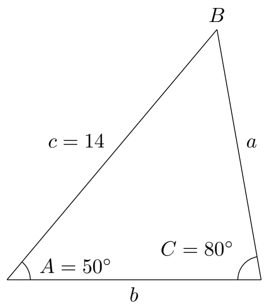
Answer. $a \approx 8.78$, $c \approx 10.76$, $C = 60°$

4. $c = 15,\ A = 65°,\ B = 45°$



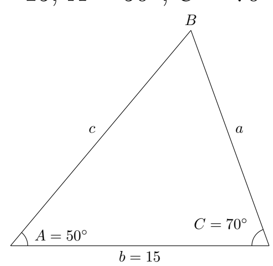
Answer. $a \approx 14.47$, $b \approx 11.29$, $C = 70°$

5. $c = 14,\ A = 50°,\ C = 80°$



Answer. $a \approx 10.89$, $b \approx 10.89$, $B = 50°$

6. $b = 15,\ A = 50°,\ C = 70°$



Answer. $a \approx 13.27$, $c \approx 16.28$, $B = 60°$

Exercise Group. Use the Law of Sines to solve each triangle. Express angles in degrees rounded to one decimal and sides rounded to two decimals.

7. $a = 8,\ A = 50°,\ C = 80°$

Answer. $b = 8$, $c \approx 10.28$, $B = 50°$

8. $b = 12,\ B = 60°,\ C = 30°$

Answer. $a \approx 13.86$, $c \approx 6.93$, $A = 90°$

9. $c = 10,\ A = 40°,\ C = 70°$

Answer. $a \approx 6.84$, $b = 10$, $B = 70°$

10. $a = 15,\ A = 40°,\ B = 100°$

Answer. $b \approx 22.98$, $c = 15$, $C = 40°$

11. $b = 20,\ B = 80°,\ A = 20°$

Answer. $a \approx 6.94$, $c = 20$, $C = 80°$

12. $a = 8,\ A = 40°,\ B = 75°$

Answer. $b \approx 12.02$, $c \approx 11.28$, $C = 65°$

13. $b = 8,\ A = 30°,\ C = 50°$

Answer. $a \approx 4.06$, $c \approx 6.22$, $B = 100°$

14. $c = 15,\ A = 70°,\ B = 45°$

Answer. $a \approx 15.55$, $b \approx 11.70$, $C = 65°$

15. $a = 10,\ B = 60°,\ C = 70°$

Answer. $b \approx 11.31$, $c \approx 12.27$, $A = 50°$

16. $b = 14,\ A = 45°,\ C = 80°$

Answer. $a \approx 12.09$, $c \approx 16.83$, $B = 55°$

17. $c = 20,\ A = 60°,\ B = 75°$

Answer. $a \approx 24.49$, $b \approx 27.32$, $C = 45°$

18. $c = 18,\ A = 70°,\ B = 50°$

Answer. $a \approx 19.53$, $b \approx 15.92$, $C = 60°$

Exercise Group. For each problem, determine the relevant case from Table 4.1.7 and whether it results one triangle, two triangles, or none. Then solve using the Law of Sines, with angles in degrees rounded to one decimal place and sides to two decimals.

19. $c = 5$, $b = 4$, $C = 120°$

Answer. Case 6; one triangle; $a \approx 1.61$, $A \approx 16.1°$, $B \approx 43.9°$

20. $b = 2$, $c = 7$, $B = 30°$

Answer. Case 1; no triangles

21. $a = 8$, $c = 6$, $A = 45°$

Answer. Case 4; one triangle; $b \approx 11.02$, $B \approx 103.0°$, $C \approx 32.0°$

22. $a = 13$, $b = 12$; $B = 46°$

Answer. Case 3; two triangles; $c_1 \approx 16.55$, $A_1 \approx 51.2°$, $C_1 \approx 82.8°$; $c_2 \approx 1.51$, $A_2 \approx 128.8°$, $C_2 \approx 5.2°$

23. $a = \sqrt{3}$, $c = 2$, $A = 60°$

Answer. Case 2; one right triangle; $b = 1$, $B = 30°$, $C = 90°$

24. $a = 7$, $c = 5$, $C = 30°$

Answer. Case 3; two triangles; $b_1 \approx 9.63$, $A_1 \approx 44.4°$, $B_1 \approx 105.6°$; $b_2 \approx 2.49$, $A_2 \approx 135.6°$, $B_2 \approx 14.4°$

25. $b = 14$, $c = 8$, $C = 115°$

Answer. Case 5; no triangles

26. $a = 4$, $b = 8$, $A = 30°$

Answer. Case 2; one right triangle; $c \approx 6.93$, $B = 90°$, $C = 60°$

27. $b = 12$, $c = 4$, $C = 55°$

Answer. Case 1; no triangles

28. $a = 7$, $b = 10$, $B = 108°$

Answer. Case 6; one triangle; $c \approx 5.30$, $A \approx 41.7°$, $C \approx 30.3°$

29. $a = 4$, $c = 5$, $A = 100°$

Answer. Case 5; no triangles

30. $b = 7$, $c = 4$, $B = 35°$

Answer. Case 4, one triangle; $a \approx 9.89$, $A \approx 125.9°$, $C \approx 19.1°$

Exercise Group. Find the area of each triangle. Round your answer to the nearest tenth.

31. $a = 8$, $b = 12$, $C = 60°$.

Answer. 41.6

32. $b = 10$, $c = 15$, $A = 45°$.

Answer. 53.0

33. $c = 6$, $a = 9$, $B = 30°$.

Answer. 13.5

34. $a = 5$, $c = 8$, $B = 45°$.

Answer. 14.1

35. $b = 7$, $c = 10$, $A = 60°$.

Answer. 30.3

36. $a = 12$, $b = 15$, $C = 75°$.

Answer. 86.9

37. $b = 6$, $c = 8$, $A = 30°$.

Answer. 12.0

38. $a = 9$, $c = 12$, $B = 60°$.

Answer. 46.8

39. A waʻa is sailing in the direction Hikina (east) at 5 knots and spots an island in the house Lā Koʻolau (11.25° above due east). Six hours later (30 nm) the navigator measures the island in the house Manu Hoʻolua (45° above due west).

(a) How far is the waʻa from the island at the time of its first measurement, rounded to the nearest nautical mile?

Answer. 26 nm

(b) How far is the waʻa from the island at the time of its second measurement, rounded to the nearest nautical mile?

Answer. 7 nm

(c) How close did the wa‘a get to the island, rounded to the nearest nautical mile?

Hint. Measure the altitude.

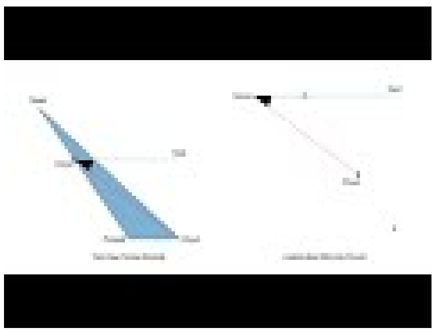
Answer. 5 nm

4.2 The Law of Cosines

For centuries, Micronesian navigators have sailed vast distances between islands, relying on ancestral knowledge passed down through generations. Using clues to determine their speed and direction, these navigators construct a mental map of their journey, viewing the canoe as stationary compared to the ever-shifting backdrop of movable islands.

Before embarking on a voyage, navigators know relative positions of islands to the canoes at the start, along their route, and at the end. As the journey unfolds, the navigator's perspective from the stationary canoe will figuratively "move islands" in their mental map of the voyage. This dynamic process, known as **etak**, allows navigators to gauge their progress and anticipate their destination even before sighting land.

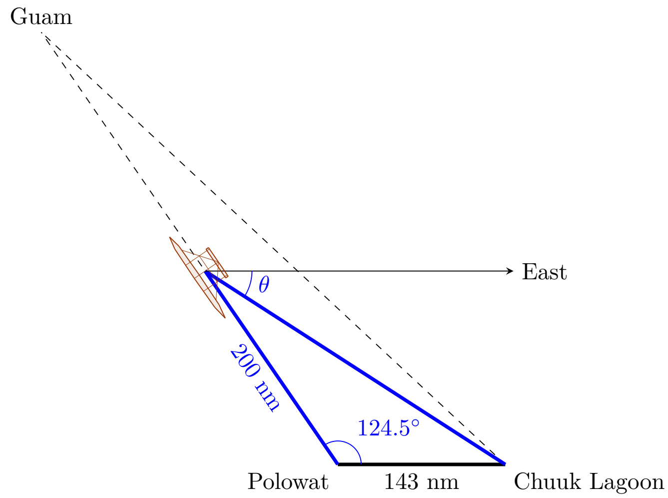
For example, consider the voyage from Polowat to Guam. Initially, Chuuk Lagoon lies due east in Tan Mailap (see Figure 1.1.2). However, as the journey progresses, Chuuk Lagoon gradually shifts southeastward, aligning with Tan Tumur at approximately 42.5° south of east at the end of the voyage at Guam. This process of moving islands is illustrated in Figure 4.2.1.



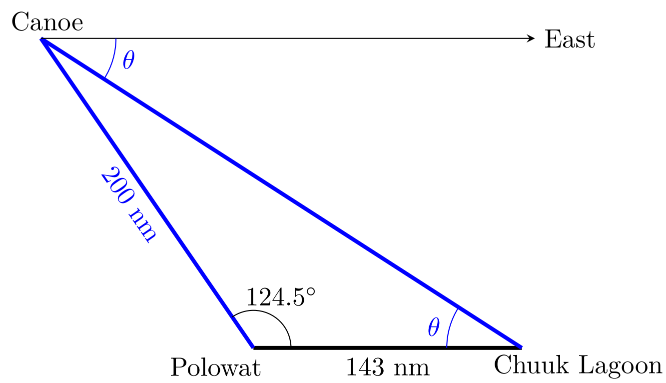
Standalone

Figure 4.2.1 In Micronesian navigation, the concept of *etak* is illustrated as islands appear to move relative to the stationary canoe. During a voyage from Polowat to Guam, Chuuk Lagoon transitions from an eastern to southeastern position, demonstrating the navigator's perspective where islands are perceived to move while the canoe remains still. If you are viewing the PDF or a printed copy, you can scan the QR code or follow the "Standalone" link to watch the video online.

Suppose the navigator is 200 nautical miles into their journey. What is θ, the angle the navigator moved Chuuk Lagoon relative to due east? Assume Chuuk is 143 nm east of Polowat and the angle formed between Chuuk Lagoon, Polowat, and Guam is 124.5°. We can illustrate what is happening below:



Since the line from Polowat to Chuuk Lagoon and the line due east from the canoe's current position are parallel, the angle θ that Chuuk Lagoon has shifted below east is equivalent to the angle formed from Polowat to Chuuk Lagoon to the canoe by the **alternate interior angles** of parallel lines in geometry. This is shown in the figure below, where the lines in blue change, depending on the position of the canoe, but the distance between Polowat and Chuuk Lagoon, represented in black, remains the same.

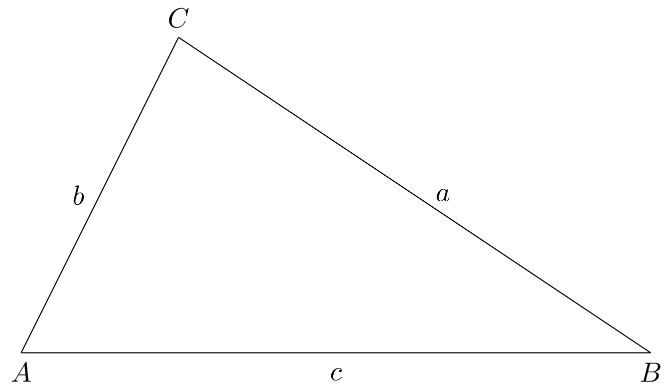


This now creates a triangle where two sides (the distances between Chuuk Lagoon and Polowat, and between Polowat and the canoe) and the included angle (from Chuuk Lagoon to Polowat to the canoe) are known. This triangular configuration is a side-angle-side (SAS) scenario. Unfortunately, we cannot apply the Law of Sines to solve this oblique triangle. In this section, we learn the principles of the Law of Cosines and how to use it to solve triangles that are side-side-side (SSS) and side-angle-side (SAS), as well as calculating the areas of triangles with only their sides given.

4.2.1 Law of Cosines

Theorem 4.2.2 Law of Cosines. *Given a triangle with sides a, b, c and opposite angles A, B, C, respectively,*

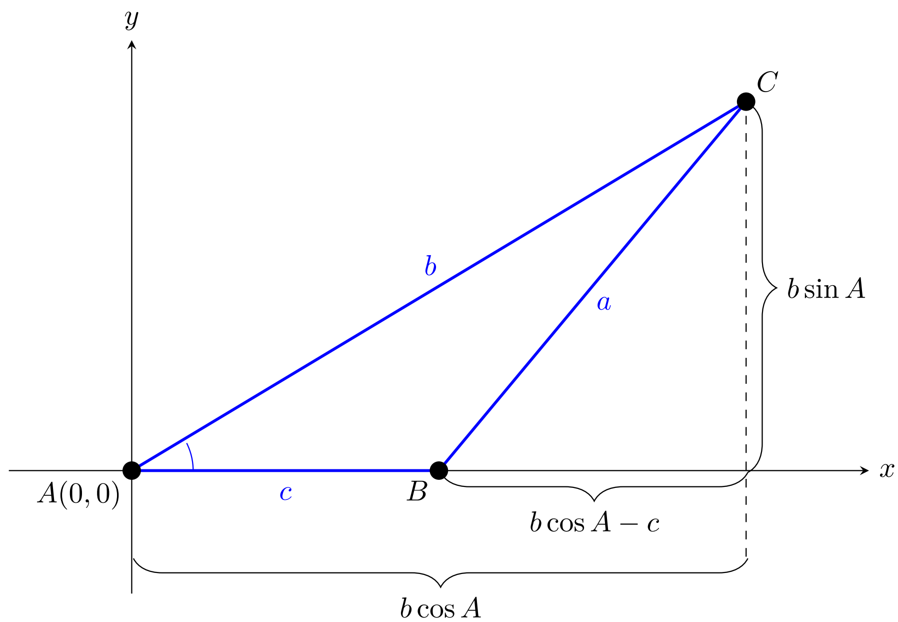
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$



Thus, in any triangle, the square of one side equals the sum of the squares of the other two sides, minus twice their product times the cosine of their included angle.

Note: If the triangle is a right triangle, say $C = 90°$, then since $\cos 90° = 0$, the Law of Cosines simplifies into the Pythagorean Theorem: $c^2 = a^2 + b^2$.

Proof. To prove the first part, we will place a triangle with A at the origin and B on the x-axis, as shown below.



Using right triangle trigonometry, the coordinates of C can be written as $(b\cos A, b\sin A)$. Using the Pythagorean Theorem, we get

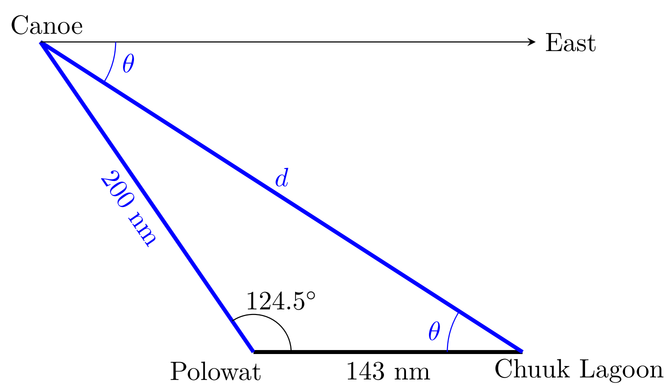
$$\begin{aligned} a^2 &= (b\cos A - c)^2 + (b\sin A)^2 \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A \\ &= b^2(1) + c^2 - 2bc\cos A \\ &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

where we used the identity $\cos^2 A + \sin^2 A = 1$. The other two parts of the Law of Cosines can be proved using a similar argument. ■

4.2.2 Using Law of Cosines (SAS)

We are now ready to revisit the problem posed at the start of this section.

Example 4.2.3 Using Law of Cosines (SAS): Etak - Moving Islands. A canoe is 200 nautical miles into a journey from Polowat to Guam. If Chuuk Lagoon is 143 nm east of Polowat and the angle formed between Chuuk Lagoon, Polowat, and Guam is 124.5°, what is θ, the angle the navigator moved Chuuk Lagoon relative to due east? First, we note that the angle formed between the canoe, Chuuk Lagoon, and Polowat is also θ, giving us a SAS triangle. If we denote the side of the triangle between Chuuk Lagoon and the canoe as d, we have the following triangle:



Solution. Since the Law of Cosines states that the square of one side equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle, we get:

$$d^2 = 200^2 + 143^2 - 2 \cdot 200 \cdot 143 \cos 124.5°.$$

Taking the square root we get:

$$d = \sqrt{200^2 + 143^2 - 2 \cdot 200 \cdot 143 \cos 124.5°} \approx 304.71.$$

Now, to find the angle θ, we can use either the Law of Sines or the Law of Cosines. If we opt for the Law of Sines, we already know side $d = 304.71$, the side between Polowat and Chuuk Lagoon at 143 nm, and the angle between the canoe, Polowat, and Chuuk Lagoon at 124.5°. This combination results in a side-side-angle (SSA) triangle. Given that our angle is obtuse (124.5°) and the side opposite it is greater than the side adjacent to the angle (i.e., $304.71 > 143$), we fall under Case 6 of Table 4.1.7, which gives us one triangle. We will choose the Law of Sines since it requires fewer steps to isolate the angle θ:

$$\frac{\sin 124.5°}{304.71} = \frac{\sin \theta}{200}.$$

Isolating $\sin \theta$:

$$\sin \theta = 200 \cdot \frac{\sin 124.5°}{304.71}.$$

Finally, to solve for θ take the inverse sine of both sides to get:

$$\theta = \sin^{-1}\left(200 \cdot \frac{\sin 124.5°}{304.71}\right) \approx 32.7°.$$

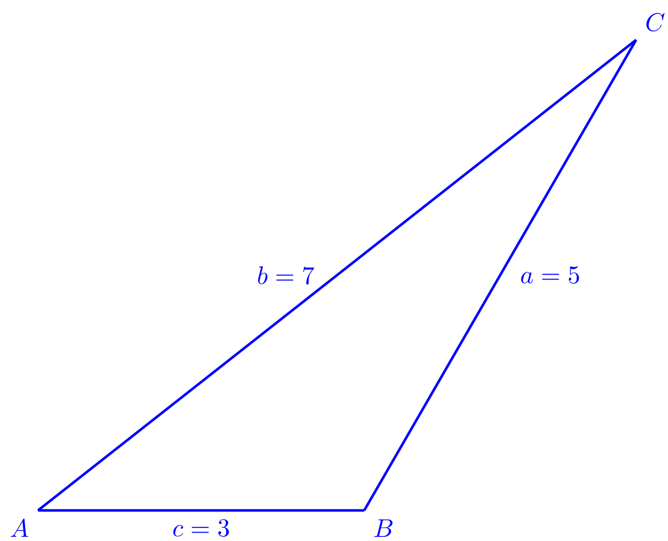
Thus, 200 nm into their voyage, the navigator has moved Chuuk Lagoon 32.7° south of due east to Tan Harapwel. □

Remark 4.2.4 Steps for solving SAS triangles. In general, when solving SAS triangles:

1. Use the Law of Cosines to solve for the remaining side.
2. To find the second angle, use either the Law of Cosines or Law of Sines.
   - Using the Law of Sines may offer a simpler method for finding the angle. Be cautious of the SSA triangle (side-side-angle), which uses the ambiguous case. Refer to Table 4.1.7 to identify the applicable case. If we encounter Case 3 (two triangles), check if the angle we need to find is acute ($< 90°$) or obtuse ($> 90°$). If it's obtuse, subtract the angle found using the Law of Sines from $180°$.
   - Using the Law of Cosines does not require verification of whether the angle is acute or obtuse, but it involves more steps to isolate the angle.
3. Use the fact that the sum of the three angles in a triangle equals $180°$ to find the third angle.

4.2.3 Using Law of Cosines (SSS)

Example 4.2.5 Using Law of Cosines (SSS). If the sides of a triangle are $a = 5$, $b = 7$, and $c = 3$, find the angles.



Solution. To find A, we begin by using the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Isolating the term with angle A we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7^2 + 3^2 - 5^2}{2 \cdot 7 \cdot 3} = \frac{33}{42}.$$

Taking the inverse cosine we get

$$A = \cos^{-1}\left(\frac{33}{42}\right) \approx 38.2°.$$

To find B, we can use either the Law of Sines or Law of Cosines. However, to avoid the ambiguous case with the Law of Sines, we will continue with the

Law of Cosines and get:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 3^2 - 7^2}{2 \cdot 5 \cdot 3} = -\frac{15}{30} = -\frac{1}{2}.$$

Taking the inverse cosine we get

$$B = \cos^{-1}\left(-\frac{1}{2}\right) = 120°.$$

To find the last angle, C, we can use the fact that the angles in a triangle sum to 180°:

$$C = 180° - A - B \approx 180° - 38.2° - 120° = 21.8°.$$

□

Remark 4.2.6 Steps for solving SSS triangles. When solving SSS triangles, the process is similar to that of solving SAS triangles, with a slight variation. Instead of solving for a side in Step 1, we solve for an angle. Steps 2 and 3 remain the same:

1. Use the Law of Cosines to solve for the first angle.
2. To find the second angle, use either the Law of Cosines or Law of Sines.
   - Using the Law of Sines may provide a simpler method for finding the angle. Be cautious of the SSA triangle (side-side-angle), which can lead to the ambiguous case. Refer to Table 4.1.7 to determine our case. If we encounter Case 3 (two triangles), verify if the angle we need to find is acute ($< 90°$) or obtuse ($> 90°$). For obtuse angles, subtract the angle found using the Law of Sines from 180°.
   - Using the Law of Cosines does not require verification of whether the angle is acute or obtuse, but it involves more steps to isolate the angle.
3. Use the fact that the sum of the three angles in a triangle equals 180° to find the third angle.

4.2.4 Area of an oblique triangle

The Greek mathematician Heron of Alexandria derived a formula for the area of a triangle when all three sides are given. Although a more modern derivation for this formula involves the Law of Cosines, it is still known as Heron's Formula.

Theorem 4.2.7 Heron's Formula. *The area of a triangle with sides a, b, and c is*

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is half the perimeter of the triangle, also known as the ***semiperimeter****.*

Proof. We begin by solving for $\cos C$ in the Law of Cosines:

$$\begin{aligned} c^2 =& a^2 + b^2 - 2ab\cos C \\ \cos C =& \frac{a^2 + b^2 - c^2}{2ab}. \end{aligned}$$

Using this and the difference of squares, we get:

$$\begin{aligned}
1+\cos C =& 1+\frac{a^2+b^2-c^2}{2ab} \\
=& \frac{2ab+a^2+b^2-c^2}{2ab} \\
=& \frac{(a^2+2ab+b^2)-c^2}{2ab} \\
=& \frac{(a+b)^2-c^2}{2ab} \\
=& \frac{((a+b)+c)((a+b)-c)}{2ab} \\
=& \frac{(a+b+c)(a+b-c)}{2ab} \\
=& \frac{2s\cdot 2(s-c)}{2ab} \\
=& \frac{2s(s-c)}{ab},
\end{aligned}$$

where the last step comes from the fact that $2s = a+b+c$ and

$$2(s-c) = 2\left(\frac{a+b+c}{2} - c\right) = a+b+c-2c = a+b-c.$$

Similarly, we have:

$$\begin{aligned}
1-\cos C =& 1-\frac{a^2+b^2-c^2}{2ab} \\
=& \frac{2ab-a^2-b^2+c^2}{2ab} \\
=& \frac{c^2-(a^2-2ab+b^2)}{2ab} \\
=& \frac{c^2-(a-b)^2}{2ab} \\
=& \frac{(c+(a-b))(c-(a-b))}{2ab} \\
=& \frac{(a+c-b)(b+c-a)}{2ab} \\
=& \frac{2(s-b)(s-a)}{ab},
\end{aligned}$$

where the last step comes from the fact that

$$2(s-b) = 2\left(\frac{a+b+c}{2} - b\right) = a+b+c-2b = a+c-b$$

and

$$2(s-a) = 2\left(\frac{a+b+c}{2} - a\right) = a+b+c-2a = b+c-a.$$

Next, we use Definition 4.1.11 for the area of a triangle:

$$\text{Area} = \frac{1}{2}ab\sin C.$$

Squaring both sides of the equation we get:

$$(\text{Area})^2 = \frac{1}{4}a^2b^2\sin^2 C$$

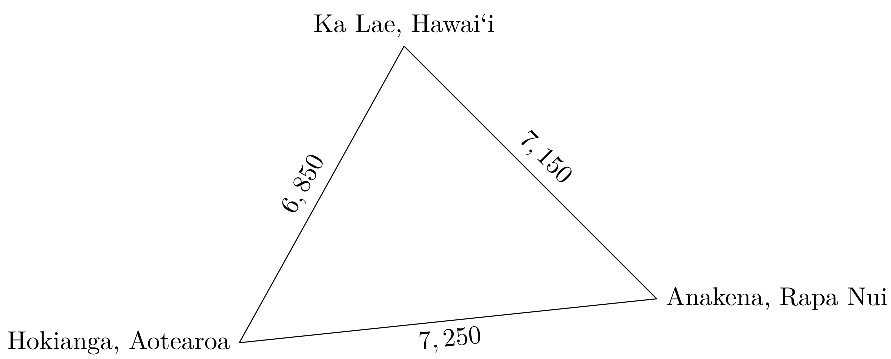
$$
\begin{aligned}
&=\frac{1}{4}a^2b^2(1-\cos^2 C)\\
&=\frac{1}{4}a^2b^2(1+\cos C)(1-\cos C)\\
&=\frac{1}{4}a^2b^2\left(\frac{2s(s-c)}{ab}\right)\left(\frac{2(s-b)(s-a)}{ab}\right)\\
&=s(s-a)(s-b)(s-c).
\end{aligned}
$$

Taking the square root of both sides of the equation gives us Heron's Formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Note that our proof differs from Heron's original proof. ■

Example 4.2.8 Polynesian Triangle. The Polynesian Triangle is an expanse in the Pacific Ocean comprising over 1,000 islands collectively known as Polynesia. Positioned at its northern corner are the Hawaiian Islands, while Aotearoa marks the southwestern corner, and Rapa Nui lies at the southeastern corner. Historically, the first settlers are believed to have arrived at Ka Lae, Hawaiʻi; Hokianga, Aotearoa; and Anakena, Rapa Nui, respectively. The distances between these pivotal locations are as follows: 6,850 km separates Ka Lae, Hawai`i from Hokianga, Aotearoa; 7,250 km span between Hokianga, Aotearoa and Anakena, Rapa Nui; and 7,150 km separate Rapa Nui from Ka Lae, Hawaiʻi. What is the area of the Polynesian Triangle, rounded to the nearest square kilometer?



Solution. Using Heron's Formula, we begin by calculating the semiperimeter

$$s = \frac{1}{2}(6,850+7,250+7,150) = 10,625.$$

Then the area created by the Polynesian Triangle is:

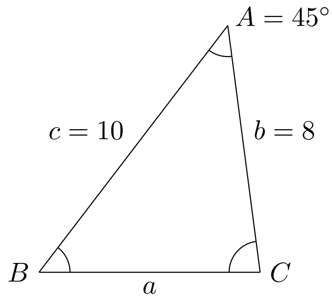
$$
\begin{aligned}
\text{Area} &= \sqrt{10,625(10,625-6,850)(10,625-7,250)(10,625-7,150)}\\
&\approx 21,688,886\text{km}^2.
\end{aligned}
$$

□

4.2.5 Exercises

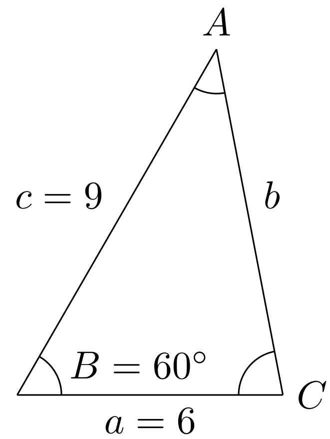
Exercise Group. Use the Law of Cosines to solve each triangle. Express angles in degrees rounded to one decimal and sides rounded to two decimals.

1. $b = 8$, $c = 10$, $A = 45°$.



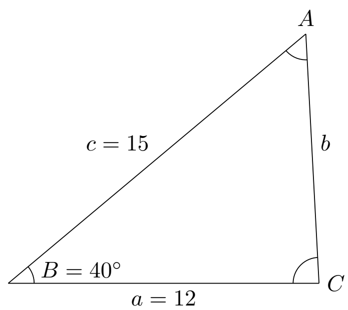
Answer. $a \approx 7.13$,
$B \approx 52.5°$, $C \approx 82.5°$

2. $a = 6$, $c = 9$, $B = 60°$.



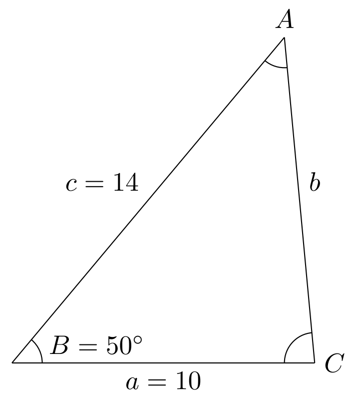
Answer. $b \approx 7.94$,
$A \approx 40.9°$, $C \approx 79.1°$

3. $a = 12$, $c = 15$, $B = 40°$.



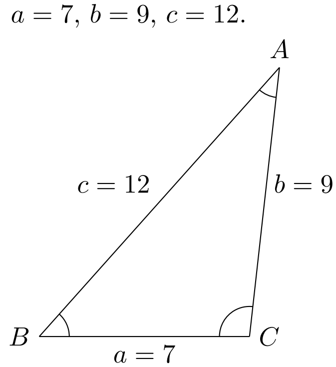
Answer. $b \approx 9.66$,
$A \approx 53.0°$, $C \approx 87.0°$

4. $a = 10$, $c = 14$, $B = 50°$.



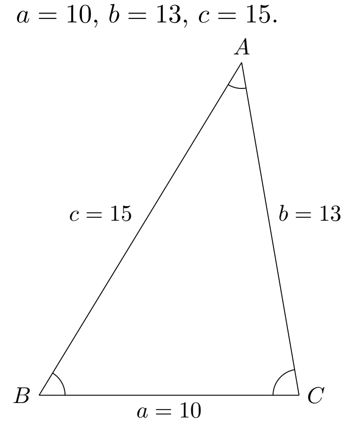
Answer. $b \approx 10.77$,
$A \approx 45.3°$, $C \approx 84.7°$

5. $a = 7$, $b = 9$, $c = 12$.



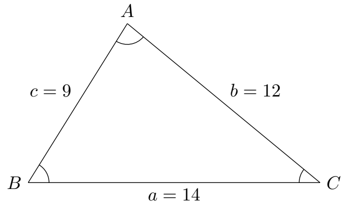
Answer. $A \approx 35.4°$,
$B \approx 48.2°$, $C \approx 96.4°$

6. $a = 10$, $b = 13$, $c = 15$.



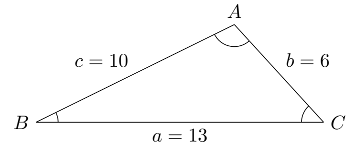
Answer. $A \approx 41.1°$,
$B \approx 58.7°$, $C \approx 80.2°$
(Rounded values may cause a slight discrepancy)

7. $a = 14$, $b = 12$, $c = 9$.



Answer. $A \approx 82.3\circ$, $B \approx 58.1°$, $C \approx 39.6°$

8. $a = 13$, $b = 6$, $c = 10$.



Answer. $A \approx 106.0°$, $B \approx 26.3°$, $C \approx 47.7°$

Exercise Group. Use the Law of Cosines to solve each triangle. Express angles in degrees rounded to one decimal and sides rounded to two decimals.

9. $a = 15$, $b = 20$, $C = 60°$.

Answer. $c \approx 18.03$, $A \approx 46.1°$, $B \approx 73.9°$

10. $b = 8$, $c = 12$, $A = 60°$.

Answer. $a \approx 10.58$, $B \approx 40.9°$, $C \approx 79.1°$

11. $a = 5$, $c = 13$, $B = 30°$.

Answer. $b \approx 9.02$, $A \approx 16.1°$, $C \approx 133.9°$

12. $a = 9$, $b = 12$, $C = 75°$.

Answer. $c \approx 13.00$, $A \approx 42.0°$, $B \approx 63.0°$

13. $b = 7$, $c = 10$, $A = 45°$.

Answer. $a \approx 7.07$, $B \approx 44.4°$, $C \approx 90.6°$

14. $a = 15$, $c = 20$, $B = 55°$.

Answer. $b \approx 16.76$, $A \approx 47.2°$, $C \approx 77.8°$ (Rounded values may cause a slight discrepancy)

15. $a = 5$, $b = 8$, $c = 9$.

Answer. $A \approx 33.6°$, $B \approx 62.2°$, $C \approx 84.2°$ (Rounded values may cause a slight discrepancy)

16. $a = 7$, $b = 21$, $c = 25$.

Answer. $A \approx 14.4°$, $B \approx 48.3°$, $C \approx 117.3°$

17. $a = 12$, $b = 16$, $c = 5$.

Answer. $A \approx 31.1°$, $B \approx 136.5°$, $C \approx 12.4°$

18. $a = 5$, $b = 2$, $c = 4$.

Answer. $A \approx 108.2°$, $B \approx 22.3°$, $C \approx 49.5°$

19. $a = 3$, $b = 7$, $c = 6$.

Answer. $A \approx 25.2°$, $B \approx 96.4°$, $C \approx 58.4°$

20. $a = 9$, $b = 10$, $c = 17$.

Answer. $A \approx 25.1°$, $B \approx 28.1°$, $C \approx 126.8°$ (Rounded values may cause a slight discrepancy)

Exercise Group. Use Heron's Formula to find the semiperimeter and area of each triangle. Round your answer to the nearest tenth

21. $a = 5$, $b = 6$, $c = 7$.

Answer. $s = 9$; $K = 14.7$

22. $a = 9$, $b = 12$, $c = 15$.

Answer. $s = 18$; $K = 54$

23. $a = 8$, $b = 10$, $c = 12$.

Answer. $s = 15$; $K = 39.7$

24. $a = 7$, $b = 9$, $c = 13$.

Answer. $s = 14.5$; $K = 30$

25. $a = 15$, $b = 20$, $c = 25$.

Answer. $s = 30$; $K = 150$

26. $a = 12$, $b = 16$, $c = 20$.

Answer. $s = 24$; $K = 96$

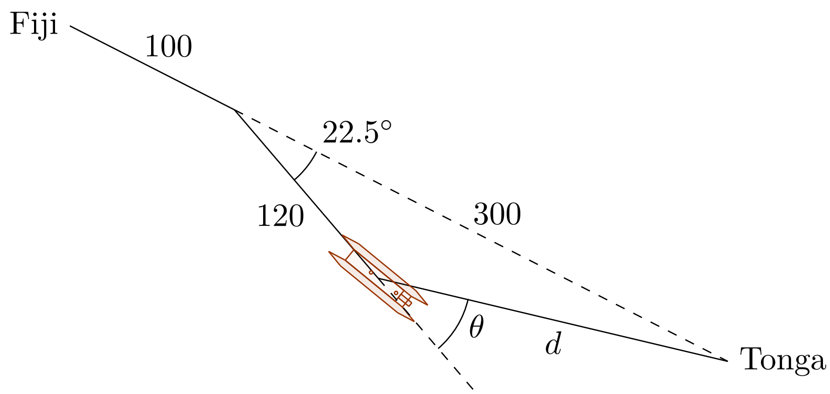
27. $a = 8$, $b = 15$, $c = 17$.

Answer. $s = 20$; $K = 60$

28. $a = 9$, $b = 40$, $c = 41$.

Answer. $s = 45$; $K = 180$

29. The Fijian canoe Uto Ni Yalo embarks on a voyage from Fiji to Tonga, covering 400 nautical miles with a heading of 117° (in the house ‘Āina Malanai). After traveling 100 nautical miles, adverse weather conditions force a deviation of two houses (22.5°) off the original course. After one day and 120 additional nautical miles, the navigator determines that a direct route to Tonga is now viable.



(a) At this point, what is the remaining distance to Tonga, d, rounded to the nearest nautical mile?

Answer. 195 NM

(b) What adjustment angle, θ, rounded to the nearest tenth of a degree, is needed to navigate to their destination?

Answer. 36.1°

(c) What was the total distance of the voyage from Fiji to Tonga, rounded to the nearest nautical mile?

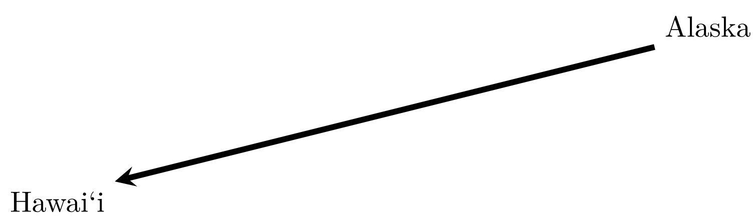
Answer. 415 NM

4.3 Geometric Vectors

Hōkūleʻa's inaugural voyage from Hawaiʻi to Tahiti in 1976 sparked a cultural resurgence in Hawaiʻi. By showcasing the effectiveness of traditional navigation techniques learned from Micronesia, it soon became clear that there was a need for constructing a waʻa kaulua (double-hulled voyaging canoe) using traditional methods and native materials. This canoe, named Hawaiʻiloa, aimed to recover knowledge and skills associated with traditional canoe building in Hawaiʻi.

However, during the years 1989-1990, an exhaustive search across Hawaiʻi revealed a significant challenge: the lack of accessible koa trees of sufficient size for constructing a canoe of this size. The search then expanded beyond Hawaiʻi. This decision was guided by historical evidence indicating that storms had transported large trees from the Pacific Coast of North America to islands like Hawaiʻi, where they were traditionally used for canoe construction. As a result, the search extended to North America in pursuit of suitable materials.

Two trees, each 200 feet high and over 400 years old, were discovered on Shelikof Island in Soda Bay, Prince of Wales Island, west of Ketchikan, Alaska. After conducting traditional tree-cutting ceremonies rooted in Hawaiian and Tlingit traditions to seek permission, both trees were felled to be used for constructing the canoe. The logs then embarked on a 2,235-nautical-mile journey to Hawaiʻi, following a heading of 256°. This journey is an example of a **vector**, a quantity possessing both **magnitude** and **direction**.



In physics, vectors are often represented with arrows, denoting their direction, while the length of the arrow signifies the vector's magnitude. Vectors play a fundamental role in describing various things in the world around us. For example, movement of objects requires a magnitude and direction, the force exerted on an object requires the magnitude of the force used and the direction of the force. In this section, we will learn how to express vectors, both geometrically and analytically as well as the various properties of vectors.

4.3.1 Geometric Vectors

Definition 4.3.1 A vector is a quantity that has both magnitude and direction. The vector $\overrightarrow{PQ}$ represents the directed segment from point P to point Q. The point P is called the **initial point**, and the point Q is called the **terminal point**. The direction of $\overrightarrow{PQ}$ is from P to Q. ♢

Definition 4.3.2 The **magnitude** of a vector $\overrightarrow{PQ}$, denoted as $\|\overrightarrow{PQ}\|$, is the distance between its initial point P and its terminal point Q. ♢

Definition 4.3.3 The angle θ represents the **direction** of a vector $\overrightarrow{PQ}$. It is the smallest positive angle measured in standard position between the positive x-axis and the vector $\overrightarrow{PQ}$, where $0° \leq \theta < 360°$. ♢

Remark 4.3.4 In physical contexts, $\overrightarrow{PQ}$ can also be interpreted as the **displacement vector**, representing the change in position from P to Q.

Remark 4.3.5 Notation. Another way to write a vector is with a lowercase boldface letter, with or without an arrow: $\mathbf{v}$ or $\overrightarrow{\mathbf{v}}$. Typically, if we are writing a vector by hand, we will want to use the arrow, since it is difficult to tell if you are writing in bold or not.

Remark 4.3.6 Calculating Magnitude and Direction. Let $\overrightarrow{PQ}$ be a vector with initial point $P = (p_x, p_y)$ and terminal point $Q = (q_x, q_y)$. The change in the x-direction (run) is denoted by $\triangle x = q_x - p_x$, and the change in the y-direction (rise) is denoted by $\triangle y = q_y - p_y$.

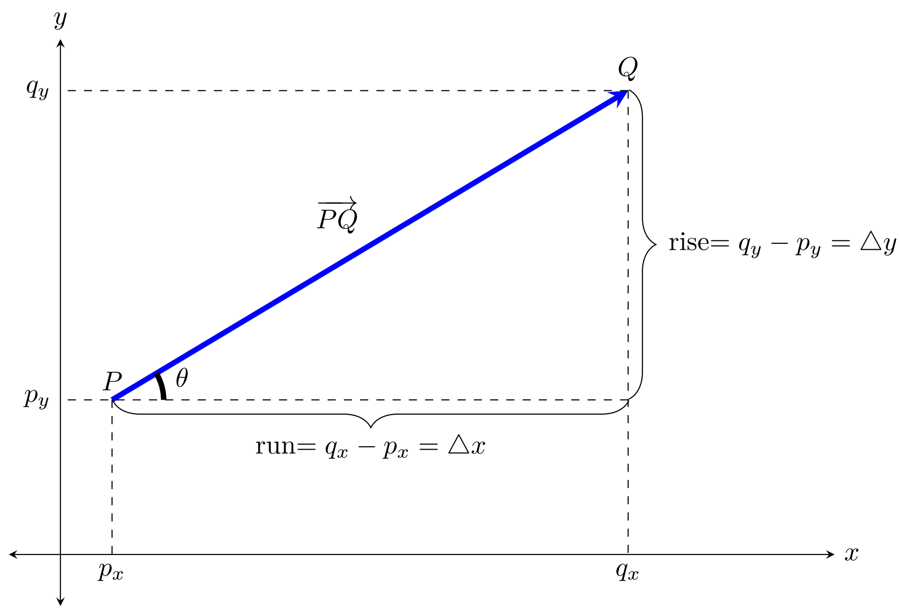
The magnitude of the vector is the length of the segment from P to Q, which can be calculated using the distance formula:

$$\|\overrightarrow{PQ}\| = \sqrt{(q_x - p_x)^2 + (q_y - p_y)^2} = \sqrt{(\triangle x)^2 + (\triangle y)^2}.$$

To determine the direction (θ) of the vector, we use the slope:

$$\text{slope} = \frac{q_y - p_y}{q_x - p_x} = \frac{\triangle y}{\triangle x}.$$

This is illustrated in the figure below.



If rise $= 0$ or run $= 0$, use the figure to determine θ. By Right Triangle Trigonometry, the tangent of the direction angle is:

$$\tan\theta = \frac{\triangle y}{\triangle x}.$$

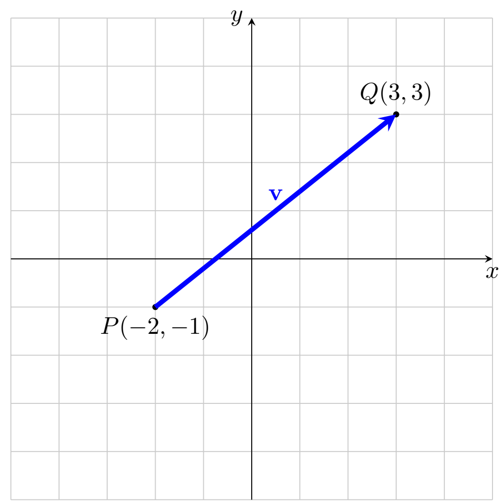
Since $-90° < \tan^{-1}\left(\frac{\triangle y}{\triangle x}\right) < 90°$, the inverse tangent function returns an angle in Quadrants I or IV. To determine the correct direction θ, consider the quadrant in which the vector lies and make the appropriate adjustments:

- If θ is in Quadrant I, then $\theta = \tan^{-1}\left(\frac{\triangle y}{\triangle x}\right)$.
- If θ is in Quadrant II or III, then $\theta = \tan^{-1}\left(\frac{\triangle y}{\triangle x}\right) + 180°$.
- If θ is in Quadrant IV, then $\theta = \tan^{-1}\left(\frac{\triangle y}{\triangle x}\right) + 360°$.

Example 4.3.7 Let $\mathbf{v}$ be the vector with initial point $P = (-2, -1)$ and terminal point $Q = (3, 3)$.

(a) Draw the vector $\mathbf{v}$.

Solution.



(b) Calculate the magnitude of $\mathbf{v}$, $\|\mathbf{v}\|$.

Solution. The magnitude of a vector is the length of the vector. From the distance formula, we have

$$\|\mathbf{v}\| = \sqrt{(3-(-2))^2 + (3-(-1))^2}.$$

(c) Calculate the direction of $\mathbf{v}$, θ.

Solution. To find the direction, we have:

$$\theta = \tan^{-1}\left(\frac{3-(-1)}{3-(-2)}\right) = \tan^{-1}\left(\frac{4}{5}\right) \approx 38.7°.$$

□

Definition 4.3.8 Zero Vector. The **zero vector**, written as $\mathbf{0}$ or $\overrightarrow{\mathbf{0}}$, is a vector with zero magnitude, meaning it has no length and it has no direction.

◊

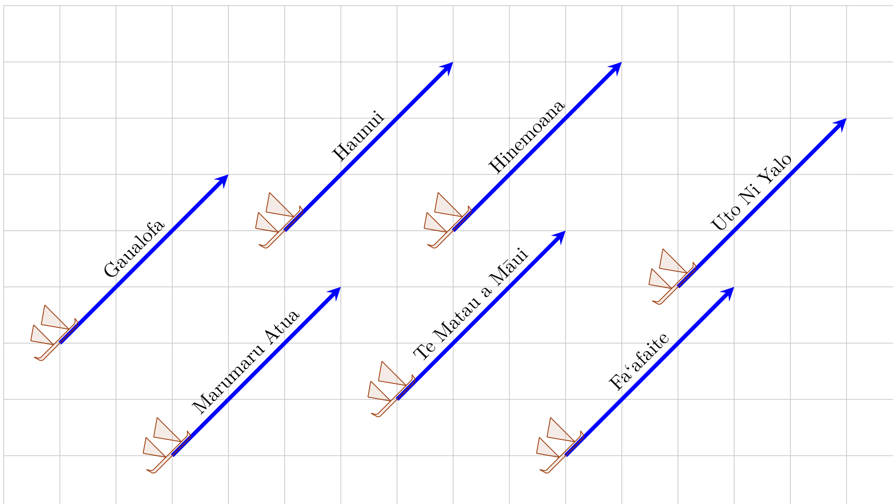
4.3.2 Equivalent Vectors

Definition 4.3.9 Equivalent Vectors. It is important to note that a vector only requires a magnitude and direction, but it does not have a unique location. Thus, as long as the magnitude and direction aren't changed, a vector may be **translated** or moved from one place to another. As a result, if two vectors $\mathbf{u}$ and $\mathbf{v}$ have the same direction and the same magnitude, then they are **equivalent**, or **equal**, written as:

$$\mathbf{u} = \mathbf{v}.$$

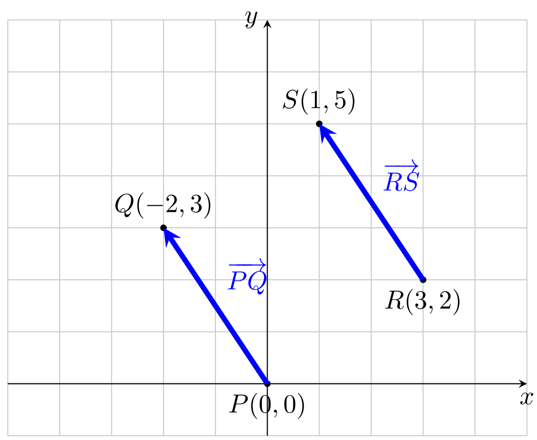
◊

Example 4.3.10 Seven canoes, reconstructed versions of the traditional double-hulled Polynesian voyaging vessels, were constructed in Aotearoa. These vessels, collectively referred to as Vaka Moana (canoes of the ocean), were individually named Marumaru Atua (Cook Islands), Gaualofa (Samoa), Te Matau a Māui (Aotearoa), Faʻafaite (Tahiti), Hinemoana (Aotearoa), Haunui (Aotearoa), and Uto Ni Yalo (Fiji). In 2011-2012, these sister canoes embarked on a voyage collectively titled Te Mana o Te Moana (The Spirit of the Ocean), covering a combined distance of 200,000 nautical miles through open waters. On average, each vaka traveled 120 nautical miles per day. Suppose one day, the vaka sailed northeast in the house Manu. The resulting vectors from each vaka are equivalent, as they all traveled in the same direction (NE) with the same magnitude (120 nm). This is depicted in the figure below.



□

Example 4.3.11 Show that vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$, shown in the figure below, are equivalent.



Solution. To show that two vectors are equivalent, we need to show they have the same magnitude and same direction. We will first calculate the magnitude of each vector.

$$\|\overrightarrow{PQ}\| = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{13}$$
$$\|\overrightarrow{RS}\| = \sqrt{(1-3)^2 + (5-2)^2} = \sqrt{13}$$

To determine the direction of each vector, we will look at their slopes. The slope of $\overrightarrow{PQ}$ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{3-0}{-2-0} = -\frac{3}{2},$$

and the slope of $\overrightarrow{RS}$ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{5-2}{1-3} = -\frac{3}{2}.$$

Given that both vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same slopes and are directed upwards and to the left, it follows that they have the same direction. Therefore, since they have the same magnitude and direction, we can conclude that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equivalent. □

4.3.3 Vector Addition

If we stand in Tahiti and face Hawai‘i, we would be looking in the house Haka Ho‘olua or north by west (NbW). When the Hōkūle‘a sails from Tahiti back home to Hawai‘i, she doesn't head directly towards Hawai‘i in the house Haka Ho‘olua. Instead, Hōkūle‘a initially sails north. Once the navigator determines that Hōkūle‘a has reached the latitude of Hawai‘i, approximately 20° N, the wa‘a then changes course to head westward towards home. This navigation technique is known as **latitude sailing**, where knowing the latitude is crucial. The navigator can determine the latitude by measuring the altitude of Hōkūpa‘a (North Star), observing the altitude of stars as they reach their highest point in the sky, or watching for pairs of stars that rise or set together.

We can represent the location of Tahiti as T, the location with the same latitude as Hawai‘i and directly north of Tahiti as L, and the location of Hawai‘i as H. Then, Hōkūle‘a's northward path can be represented by the vector $\overrightarrow{TL}$, and its westward path by the vector $\overrightarrow{LH}$. The total displacement is equivalent to sailing directly from Tahiti to Hawai‘i, represented by $\overrightarrow{TH}$. See Figure 4.3.12. We refer to the vector $\overrightarrow{TH}$ as the **sum** of the vectors $\overrightarrow{TL}$ and $\overrightarrow{LH}$, written as

$$\overrightarrow{TL} + \overrightarrow{LH} = \overrightarrow{TH}.$$

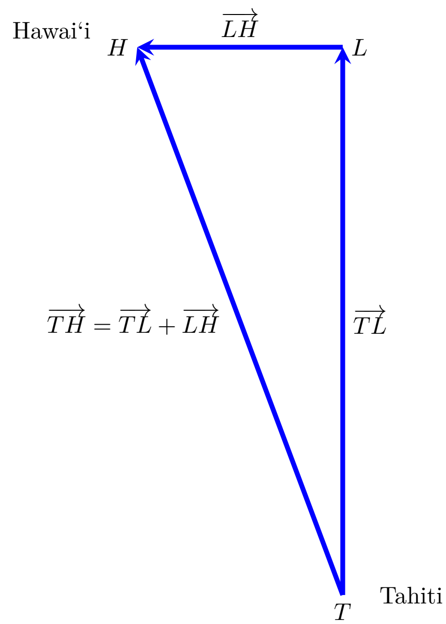


Figure 4.3.12 Combining the northward and westward vectors, similar to the navigation technique of latitude sailing, results in the direct vector from Tahiti to Hawai‘i.

Definition 4.3.13 Vector Addition. Vector addition is the process of combining two or more vectors to produce a **resultant vector**. To find the **sum** of vectors $\mathbf{u}$ and $\mathbf{v}$, we place the initial point of the second vector at the terminal point of the first vector. The vector formed from the initial point of the first vector to the terminal point of the second vector represents the sum of the two vectors, denoted by $\mathbf{u} + \mathbf{v}$. Since vectors can be translated, any two vectors can be added by shifting one vector's starting point to the endpoint of the other vector. Notice that adding vector $\mathbf{v}$ to vector $\mathbf{u}$ produces the same result as adding $\mathbf{u}$ to $\mathbf{v}$, as demonstrated below. Thus, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

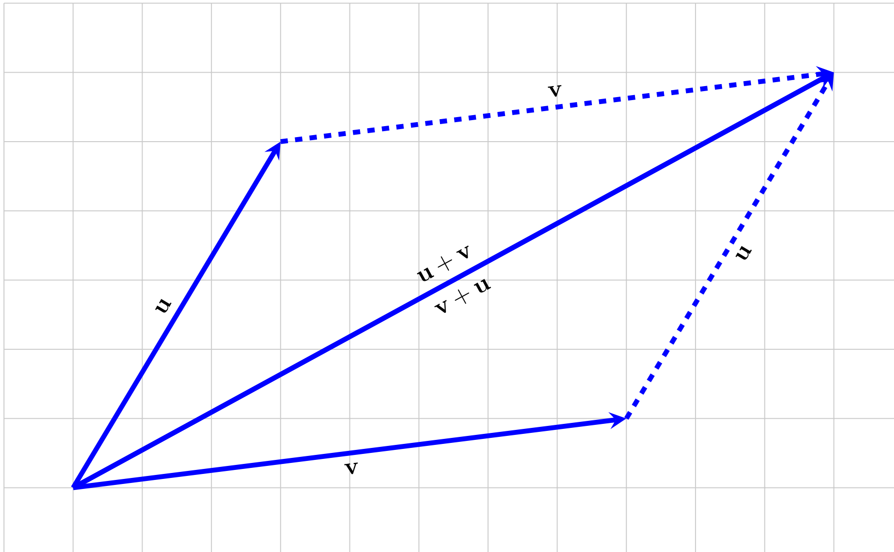
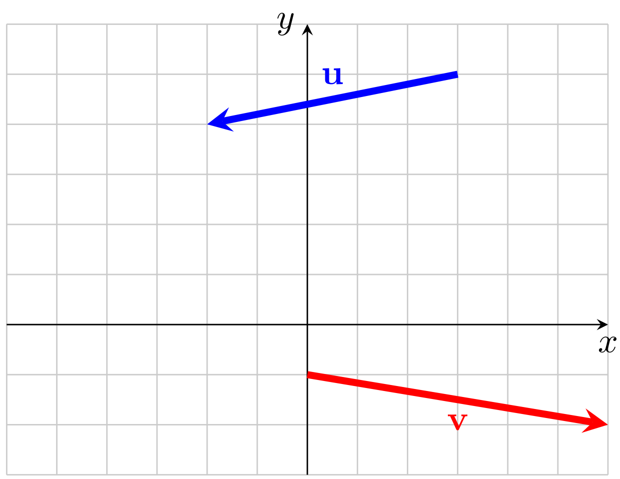


Figure 4.3.14 The vectors $\mathbf{u}$ and $\mathbf{v}$ can be added to obtain $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$, resulting in the same resultant vector. The arrangement is a parallelogram, demonstrating the commutative property of vector addition.

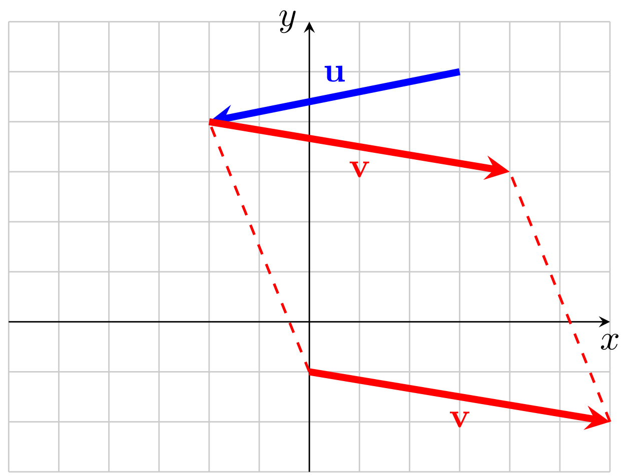
◊

Example 4.3.15 Let $\mathbf{u}$ be a vector whose initial point is at $(3, 5)$ and terminal point at $(-2, 4)$ and $\mathbf{v}$ be a vector whose initial point is at $(0, -1)$ and terminal point at $(6, -2)$. Draw the vector $\mathbf{u} + \mathbf{v}$.

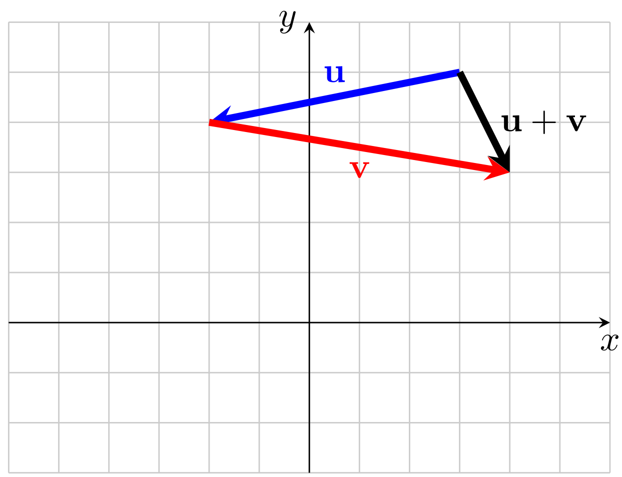
Solution. We will begin by drawing the vectors $\mathbf{u}$ and $\mathbf{v}$:



To find $\mathbf{u} + \mathbf{v}$, we will translate vector $\mathbf{v}$ so that its initial point coincides with the terminal point of $\mathbf{u}$, which is at $(-2, 4)$. The translated vector will have the same magnitude and direction as $\mathbf{v}$:



The sum of the two vectors is the vector starting from the initial point of **u** and ending at the terminal point of **v**, in this case from $(3, 5)$ to $(4, 3)$. This resultant vector is $\mathbf{u} + \mathbf{v}$:



□

4.3.4 Multiplying Vectors by a Scalar

When a wa‘a sails for two days at 5 knots, it covers a distance of 240 miles. This distance traveled is known as the **magnitude** of the vector and is represented by a real number, which we refer to as a **scalar**.

The magnitude of the vector representing a wa‘a’s path for two days will be twice that of the vector representing the path sailed for only one day. We can now define scalar multiplication of vectors.

Definition 4.3.16 Scalar Multiplication of Vectors. If c is a scalar (real number) and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ has magnitude equal to the absolute value of c times the magnitude of $\mathbf{v}$, $|c|\|\mathbf{v}\|$. Additionally:

- If $c > 0$, the scalar multiple $c\mathbf{v}$ has the same direction as $\mathbf{v}$.
- If $c < 0$, the scalar multiple $c\mathbf{v}$ has the opposite direction as $\mathbf{v}$.

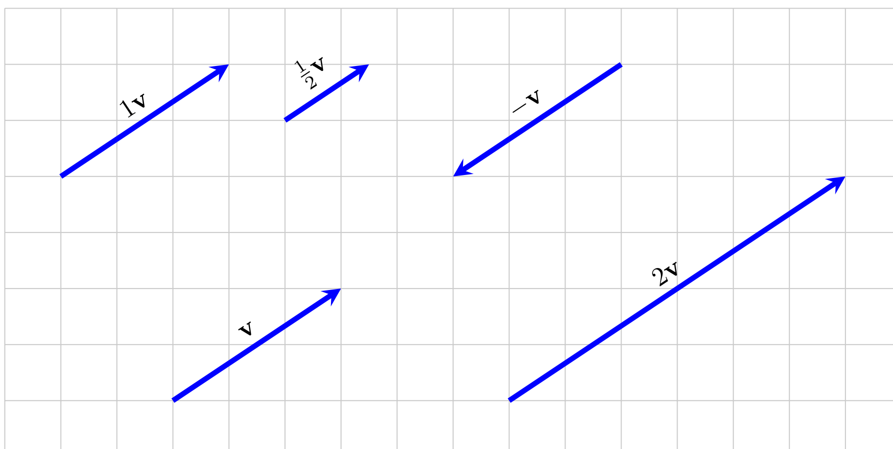
If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$. ◊

Example 4.3.17 Scalar Multiplication. To demonstrate scalar multiplication, consider a vector $\mathbf{v}$ representing the displacement of a canoe. Let's say $\mathbf{v}$ is a vector with magnitude 100 miles, pointing northeast. We can visualize this vector as representing the journey of a canoe from one point to another.

Now, let's explore scalar multiplication applied to this vector:

- $1\mathbf{v}$: This represents the same direction and magnitude as $\mathbf{v}$.
- $\frac{1}{2}\mathbf{v}$: This represents half the magnitude of $\mathbf{v}$, so it would have a magnitude of 50 miles, still pointing northeast.
- $-\mathbf{v}$: This represents the opposite direction of $\mathbf{v}$, so it would have the same magnitude of 100 miles but pointing southwest instead.
- $2\mathbf{v}$: This represents doubling the magnitude of $\mathbf{v}$, so it would have a magnitude of 200 miles, still pointing northeast.

The figure below illustrates the original vector $\mathbf{v}$ and the resulting vectors $1\mathbf{v}$, $\frac{1}{2}\mathbf{v}$, $-\mathbf{v}$, and $2\mathbf{v}$.

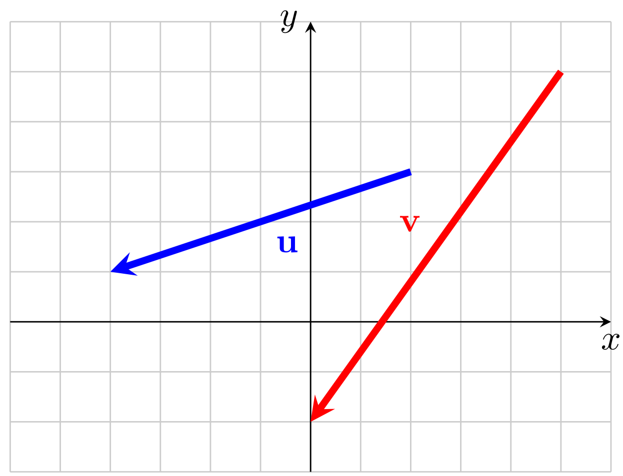


□

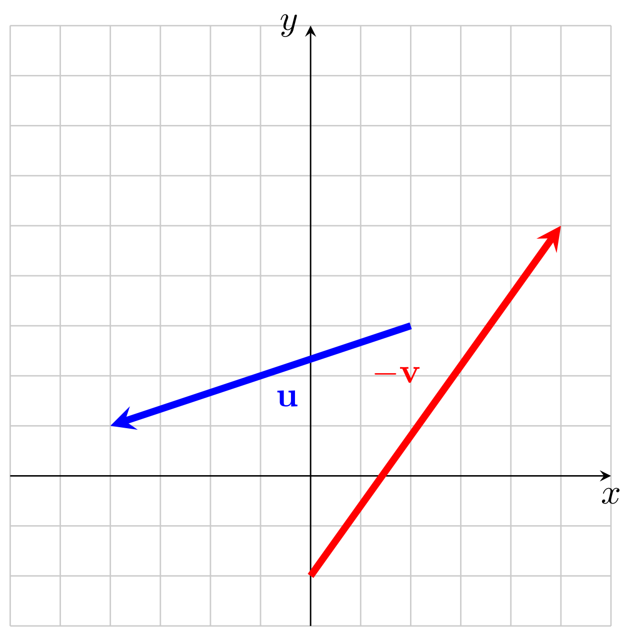
4.3.5 Vector Subtraction

Definition 4.3.18 Vector Subtraction. Vector **subtraction** is defined as the sum of one vector with the negative of another vector. In other words, $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. ◊

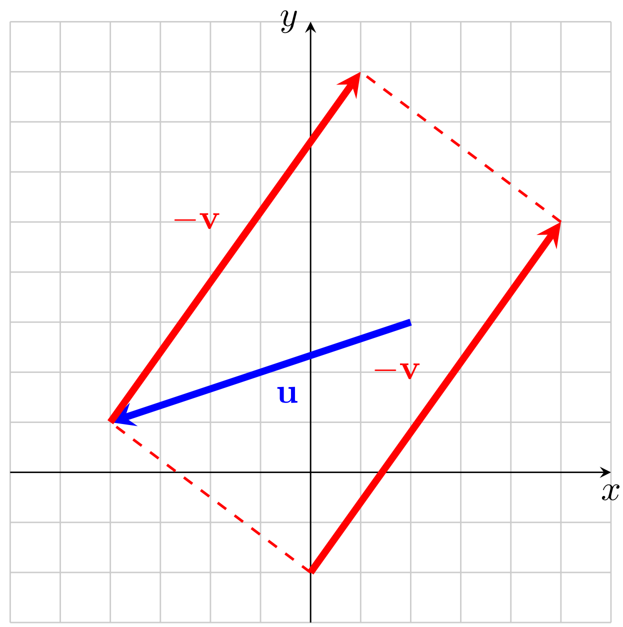
Example 4.3.19 Given vectors $\mathbf{u}$ and $\mathbf{v}$, shown in the figure below, find $\mathbf{u} - \mathbf{v}$.



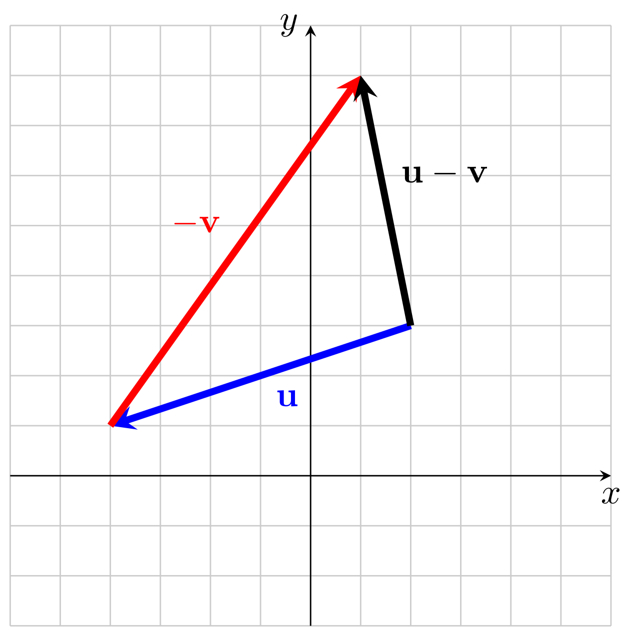
Solution. By Definition 4.3.18, finding $\mathbf{u}-\mathbf{v}$ is equivalent to finding $\mathbf{u}+(-\mathbf{v})$ so we first need to find $-\mathbf{v}$. We do this by flipping vector $\mathbf{v}$ so that it points in the opposite direction (from $(0,-2)$ to $(5,5)$):



To add $\mathbf{u}$ and $-\mathbf{v}$, we will place the initial point of $-\mathbf{v}$ at the terminal point of vector $\mathbf{u}$. This is at the point $(-4,1)$.



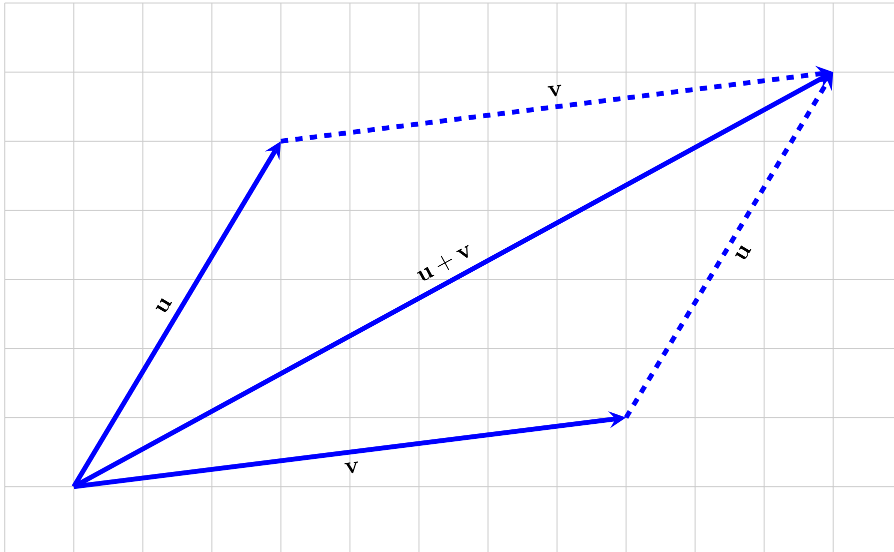
The resultant vector from the initial point of $\mathbf{u}$ to the terminal point of $-\mathbf{v}$ is the vector $\mathbf{u} - \mathbf{v}$.



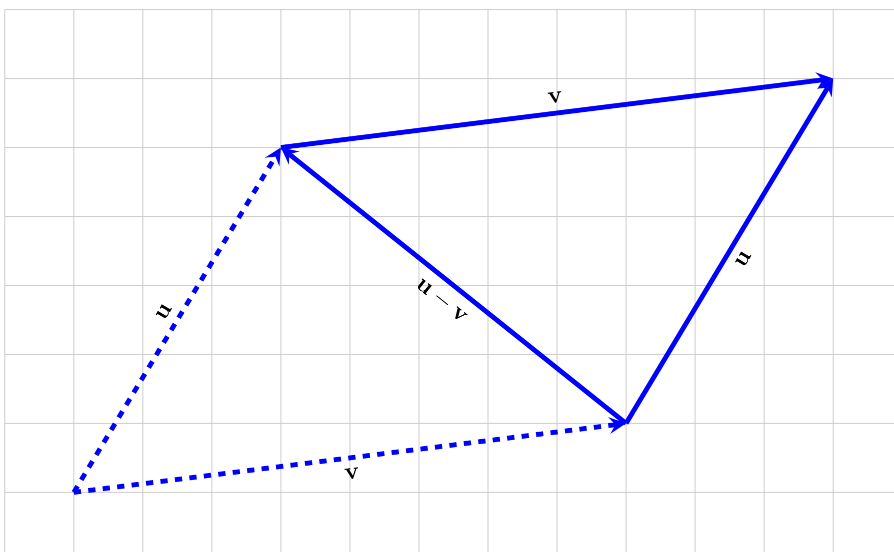
□

Remark 4.3.20 Vector Subtraction in a Parallelogram. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can visualize their sum and difference using a parallelogram. Translating one of the vectors so that their initial points coincide, we construct a parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

The *sum* of the two vectors, $\mathbf{u} + \mathbf{v}$, is represented by the diagonal of the parallelogram extending from the common initial point to the opposite corner. This diagonal shows the resultant vector when moving along $\mathbf{u}$ and then $\mathbf{v}$.



The *difference* of the two vectors, $\mathbf{u} - \mathbf{v}$, corresponds to the other diagonal of the parallelogram. This diagonal extends from the initial point of $\mathbf{u}$ to the terminal point of $-\mathbf{v}$, meaning it represents the vector obtained by adding $\mathbf{u}$ and the negation of $\mathbf{v}$.



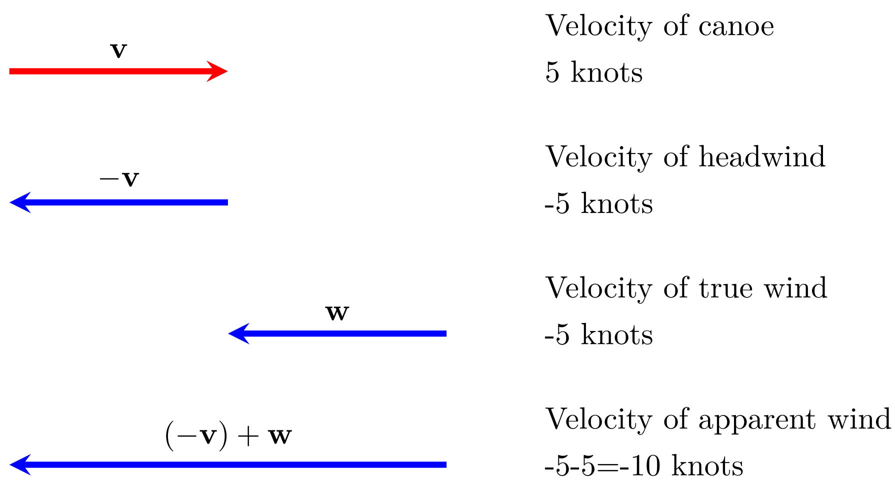
4.3.6 Velocity

Vectors can represent movement of various physical objects. For instance, the velocity of a moving wa‘a (canoe) is depicted by a vector, which indicates the direction of motion and the speed of the wa‘a. It is called the **velocity vector** and can also describe other physical phenomena such as wind and currents.

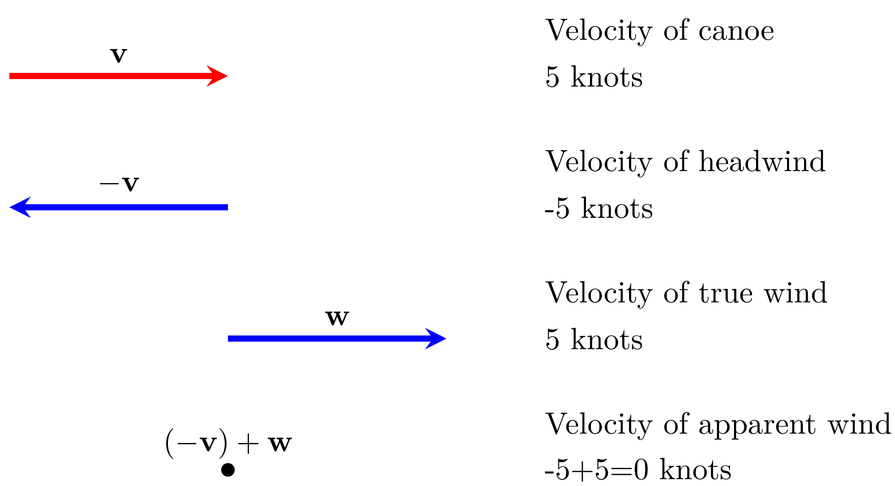
Example 4.3.21 Apparent Wind. Standing on the deck of a moving canoe, we’ll feel the wind against our face. However, this sensation is not solely due to the wind moving over the water. It is influenced by two factors: the actual wind blowing across the ocean, known as the **true wind**, and the wind created by the canoe’s own motion, called the **headwind**. The combination of these two is what we perceive as the **apparent wind**.

Headwind is similar to the resistance felt when extending our hand out of a moving car window on a still day—it results purely from our motion through the air. Mathematically, headwind is the negative of the canoe's velocity vector. The **apparent wind** is the vector sum of the **true wind** and the **headwind**.

The velocity of the **apparent wind** is determined by adding the velocities of headwind and true wind. If the true wind comes from the bow (front) of the canoe, the apparent wind increases. Conversely, if the true wind comes from the stern (back), the apparent wind decreases. For instance, sailing at 5 knots into a true wind of 5 knots would result in an apparent wind of 10 knots, as shown in the figure below.

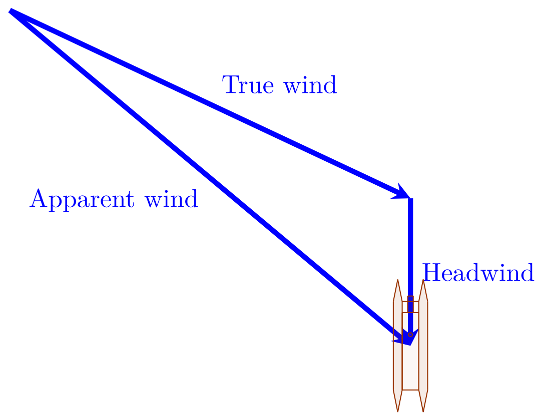


However, sailing at 5 knots with the true wind also blowing at 5 knots from behind would result in a zero apparent wind, as we would be moving at the same velocity as the wind, with no wind moving past us. This is shown in the following figure.

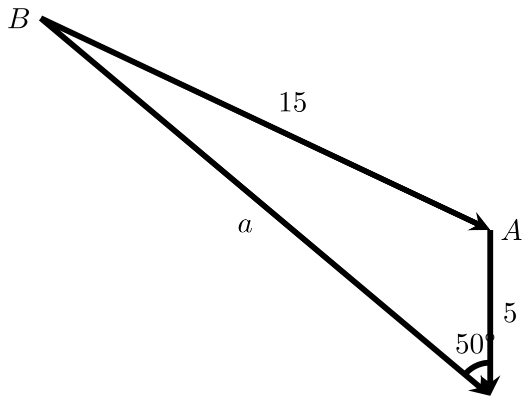


Understanding the apparent wind is crucial for sail trimming and optimizing energy from the wind. Navigators must also consider apparent wind's impact on the canoe's speed and course. □

Example 4.3.22 Apparent Wind. The waʻa Mānaiakalani is sailing off the shore of Lāhainā at a speed of 5 knots. According to the weather report, the true wind is moving at 15 knots. By turning our face into the wind so that we can feel the wind blowing evening past both sides of our face, we can determine the direction of the apparent wind, which we measure to 50° to the left of our course. What is the magnitude of the apparent wind?



Solution. We have the following triangle:



We can analyze this scenario using the Law of Sines. Given that the known angle is 50° and the side opposite to it is 15 knots (the true wind), and the side adjacent to it is 5 knots (the canoe's speed), we have a Side-Side-Angle (SSA) triangle situation, specifically Case 4, which results in only one possible triangle.

First, we find angle B using the Law of Sines:

$$\frac{\sin 50^\circ}{15} = \frac{\sin B}{5}.$$

From this, we get:

$$B = \sin^{-1}\left(\frac{5\sin 50^\circ}{15})\right) \approx 14.8^\circ.$$

Now, we can now calculate angle A:

$$A = 180^\circ - 50^\circ - 14.8^\circ = 115.2^\circ.$$

Next, to determine the magnitude of the apparent wind, we find side a using the Law of Sines:

$$\frac{a}{\sin 115.2^\circ} = \frac{15}{\sin 50^\circ}.$$

Therefore,

$$a = \sin 115.2^\circ \cdot \frac{15}{\sin 50^\circ} \approx 17.7.$$

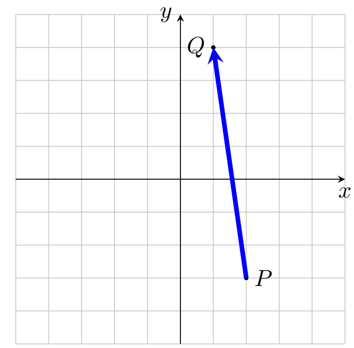
Thus, the magnitude of the apparent wind is approximately 17.7 knots. □

4.3.7 Exercises

Exercise Group. Draw the vector with initial point P and terminal point Q.

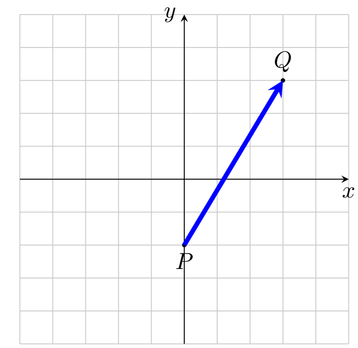
1. $P = (2, -3)$; $Q = (1, 4)$

Answer.



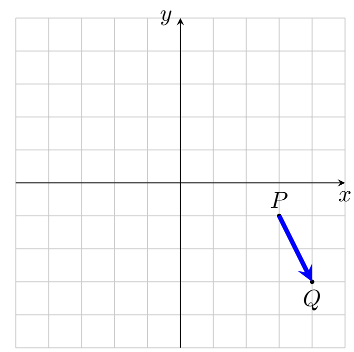
2. $P = (0, -2)$; $Q = (3, 3)$

Answer.



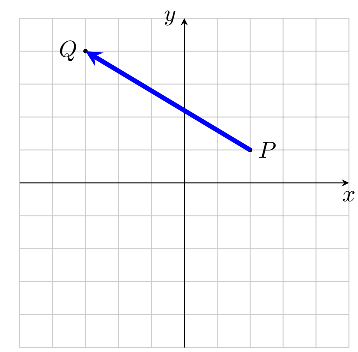
3. $P = (3, -1)$; $Q = (4, -3)$

Answer.



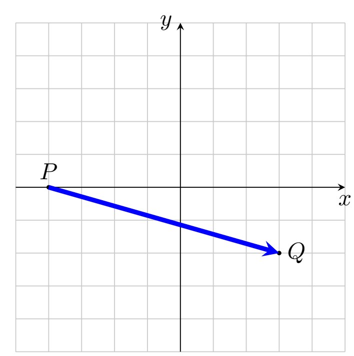
4. $P = (2, 1)$; $Q = (-3, 4)$

Answer.



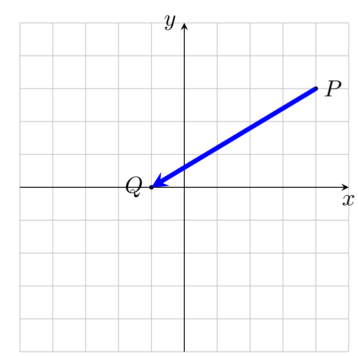
5. $P = (-4, 0)$; $Q = (3, -2)$

Answer.



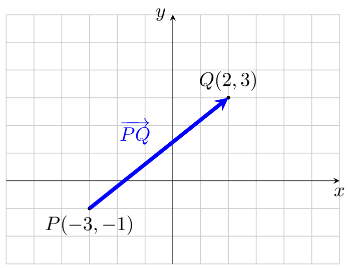
6. $P = (4, 3)$; $Q = (-1, 0)$

Answer.



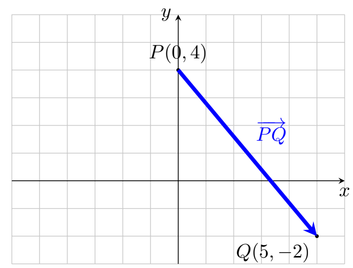
Exercise Group. Given the vector $\overrightarrow{PQ}$, where P is the initial point and Q is the terminal point, compute the magnitude $\|\overrightarrow{PQ}\|$ and the direction θ. Express the direction in degrees, rounded to one decimal place.

7.



Answer. $\sqrt{41}$, 38.7°

8.



Answer. $\sqrt{61}$, 309.8°

9. $P = (2, -4)$; $Q = (3, 2)$

Answer. $\sqrt{37}$, 80.5°

10. $P = (-2, 2)$; $Q = (0, 1)$

Answer. $\sqrt{5}$, 333.4°

11. $P = (-3, 5)$; $Q = (4, 1)$

Answer. $\sqrt{65}$, 330.3°

12. $P = (-1, -5)$; $Q = (-4, 0)$

Answer. $\sqrt{34}$, 121.0°

13. $P = (5, 4)$; $Q = (1, 3)$

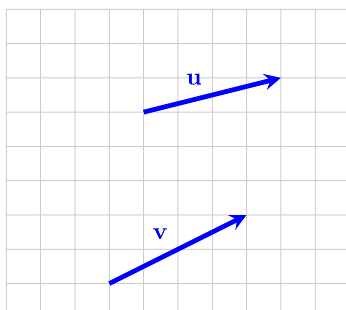
Answer. $\sqrt{17}$, 194.0°

14. $P = (-5, -1)$; $Q = (0, -3)$

Answer. $\sqrt{29}$, 338.2°

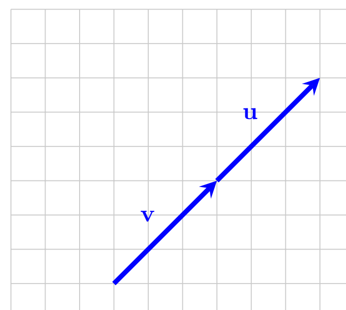
Exercise Group. Determine if the given vectors **u** and **v** are equivalent.

15.



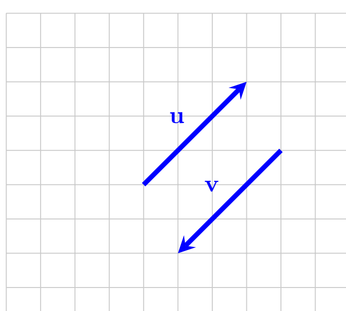
Solution. Not Equivalent

16.



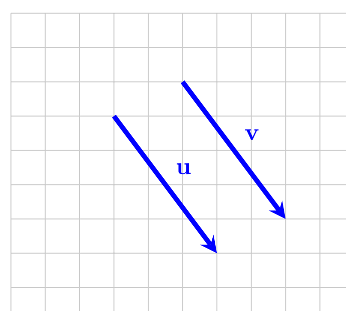
Solution. Equivalent

17.



Solution. Not Equivalent

18.



Solution. Equivalent

19. **u** has initial point at $(-1, 1)$ and terminal point at $(3, -4)$; **v** has initial point at $(0, 2)$ and terminal point at $(4, -3)$.

Answer. Equivalent

20. **u** has initial point at $(2, 1)$ and terminal point at $(-3, 2)$; **v** has initial point at $(1, -4)$ and terminal point at $(6, -3)$;.

Answer. Not Equivalent

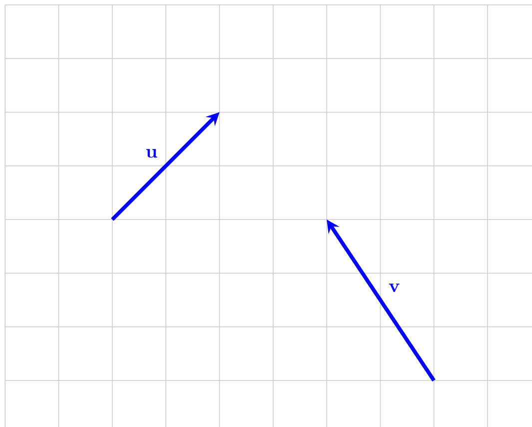
21. **u** has initial point at $(2, 0)$ and terminal point at $(4, 4)$; **v** has initial point at $(3, -2)$ and terminal point at $(-4, 1)$.

Answer. Not Equivalent

22. **u** has initial point at $(2, 4)$ and terminal point at $(0, -3)$; **v** has initial point at $(1, 2)$ and terminal point at $(-1, -5)$.

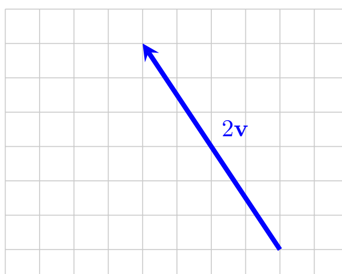
Answer. Equivalent

Exercise Group. Use the vectors **u** and **v**, shown below, to graph the following vectors:



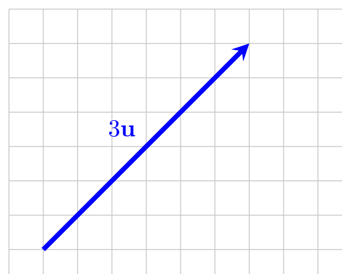
23. $2\mathbf{v}$

Answer.



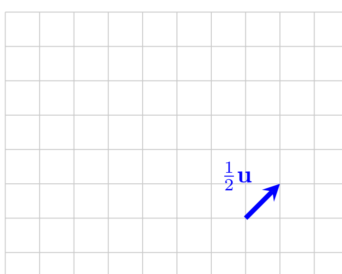
24. $3\mathbf{u}$

Answer.



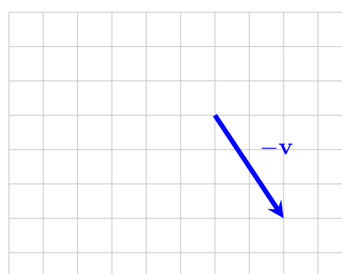
25. $\frac{1}{2}\mathbf{u}$

Answer.



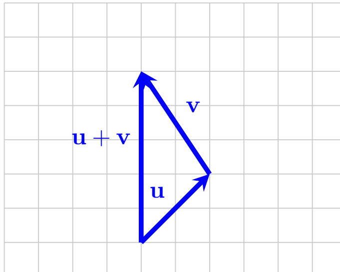
26. $-\mathbf{v}$

Answer.



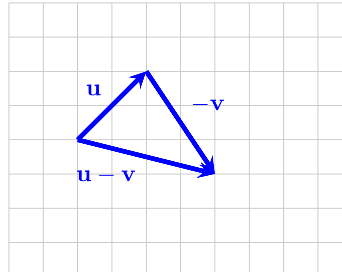
27. $\mathbf{u}+\mathbf{v}$

Answer.



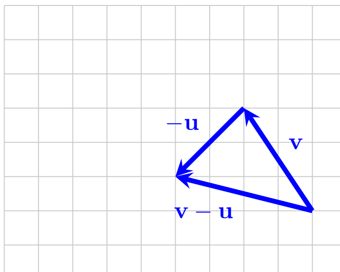
28. $\mathbf{u}-\mathbf{v}$

Answer.



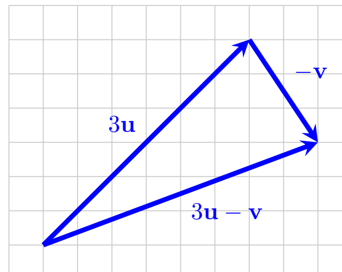
29. $\mathbf{v}-\mathbf{u}$

Answer.



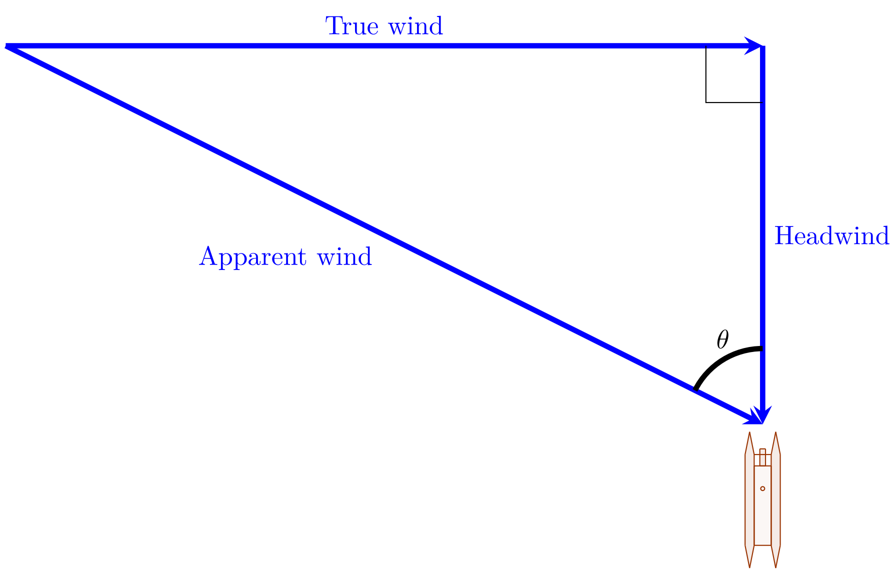
30. $3\mathbf{u}-\mathbf{v}$

Answer.

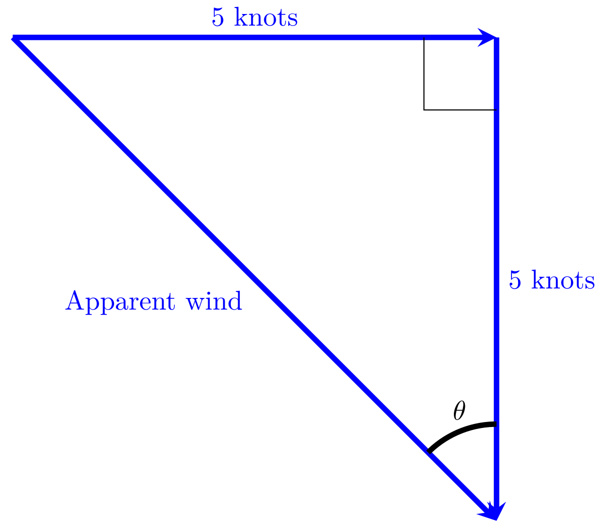


Exercise Group. Changes in the speed of the true wind can alter the apparent wind. In each scenario below, a canoe sails at 5 knots, with the true wind blowing perpendicular to the canoe, as depicted in the figure. Calculate both the direction, θ, and magnitude of the apparent wind for the given true wind speed, rounded to one decimal. Since this forms a right triangle, we use the formula:

$$\theta = \tan^{-1}\frac{\text{true wind}}{\text{headwind}}.$$

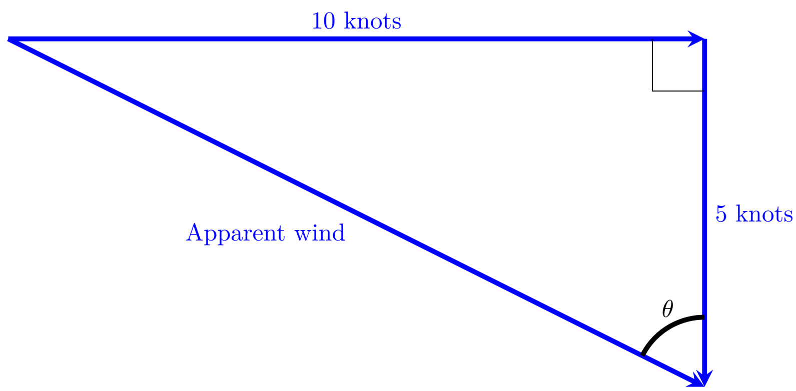


31. True wind: 5 knots



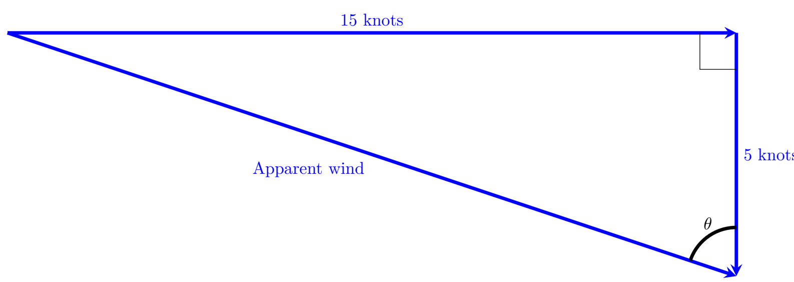
Answer. 45°; 7.1 knots

32. True wind: 10 knots



Answer. 63.4°; 11.2 knots

33. True wind: 15 knots

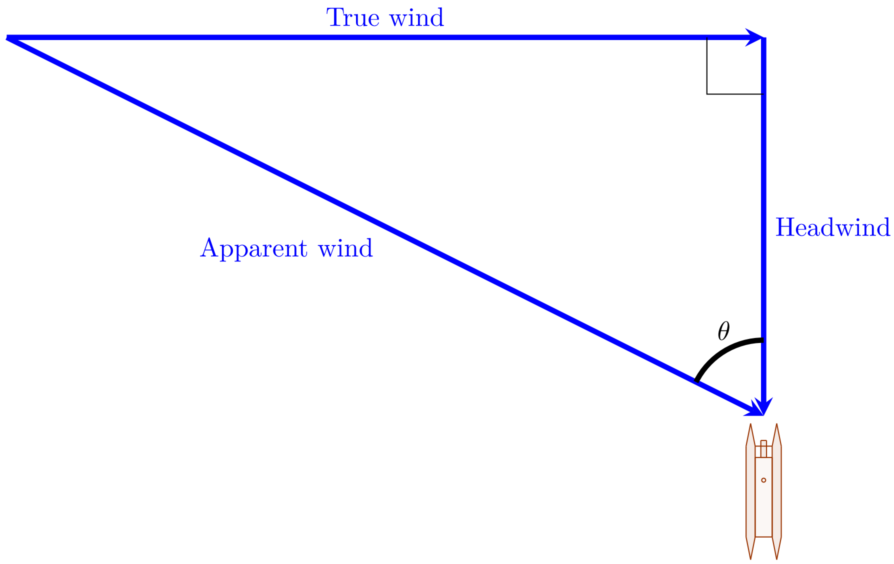


Answer. 71.6°; 15.8 knots

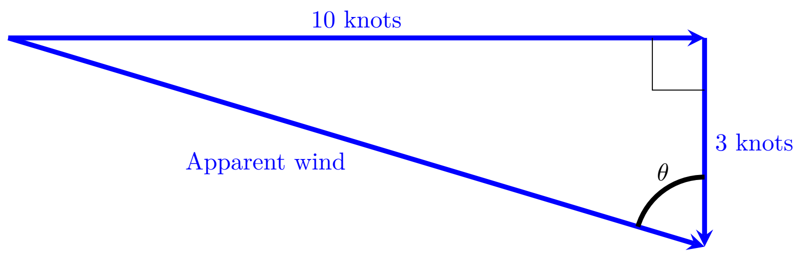
34. How does the apparent wind change as the true wind increases?

Answer. As the true wind increases, the angle and magnitude of the apparent wind increase.

Exercise Group. In addition to changes in the speed of the true wind, the apparent wind can be affected by changes in the speed of the canoe. In each scenario below, true wind blows perpendicular to the canoe at 10 knots, as depicted in the figure. Calculate both the direction, θ, and magnitude of the apparent wind for the given speed of the canoe, rounded to one decimal.

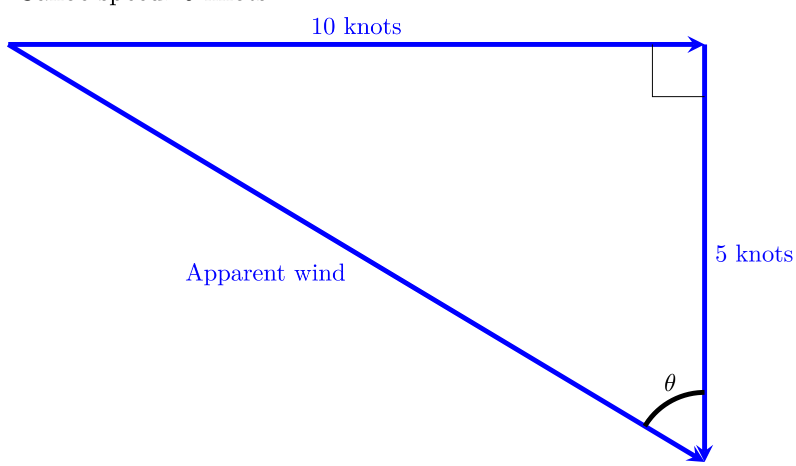


35. Canoe speed: 3 knots



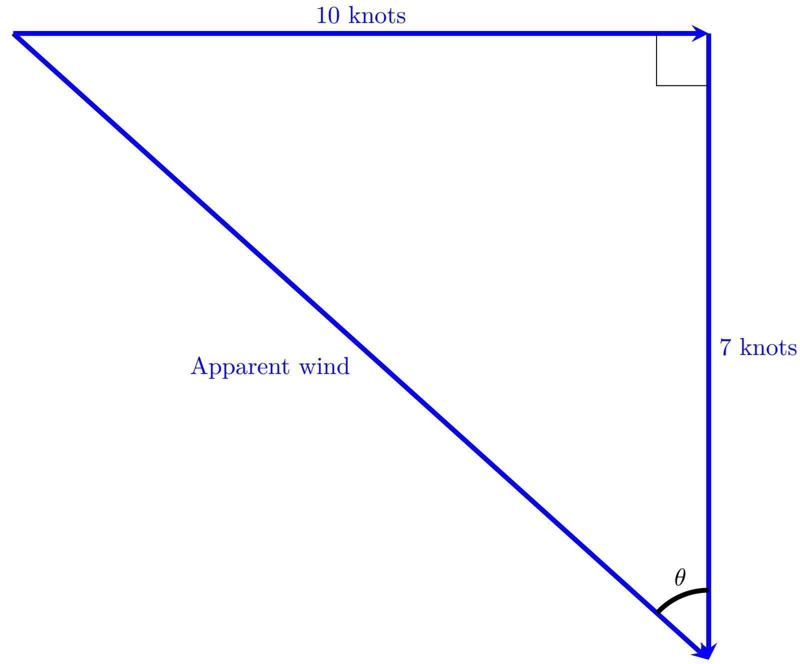
Answer. 73.3°; 10.4 knots

36. Canoe speed: 6 knots



Answer. 59.0°; 11.7 knots

37. Canoe speed: 9 knots



Answer. 48.0°; 13.5 knots

38. How does the apparent wind change as the true wind increases?

Answer. As the speed of the canoe increases, the angle decreases and the magnitude of the apparent wind increases.

4.4 Coordinate Vectors

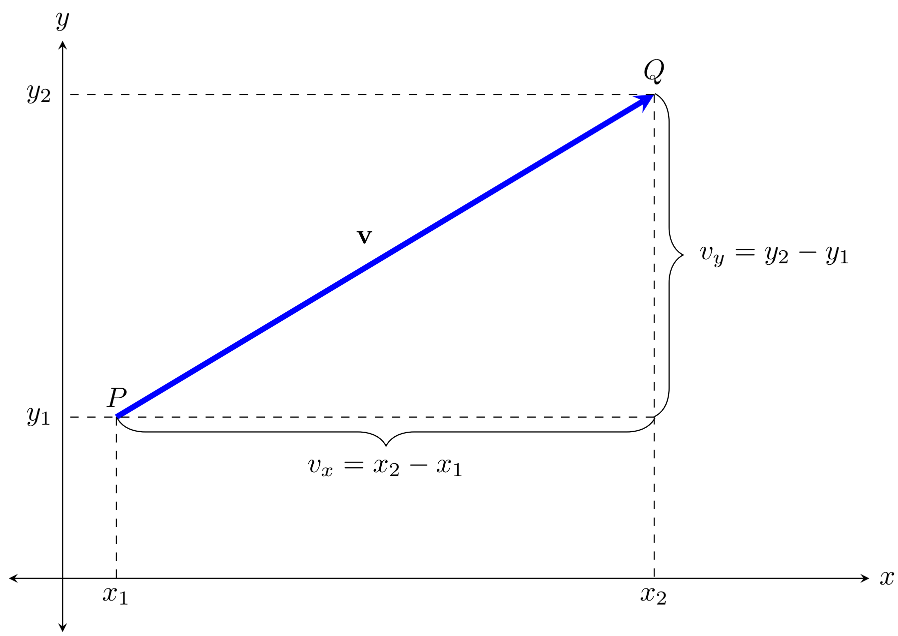
In Section 4.3, we visualized vectors as directed line segments with magnitudes and direction. Now, we will explore vectors through a coordinate approach, where we express vectors in terms of their components along the coordinate axes .

4.4.1 Vectors in Coordinates

Definition 4.4.1 Component Form of a Vector. A vector $\mathbf{v}$ with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$ can be expressed in various ways. One common representation is as the directed line segment from point P to point Q, denoted as $\overrightarrow{PQ}$. Alternatively, we can describe it using the changes in the $x-$ and $y-$ coordinates between points P and Q. These changes are referred to as the **horizontal component** and **vertical component** of the vector, denoted as v_x and v_y respectively. Mathematically, these representations are all equivalent:

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} \\ &= \langle x_2 - x_1, y_2 - y_1 \rangle \\ &= \langle v_x, v_y \rangle\end{aligned}$$

as illustrated in the following figure.



◊

Remark 4.4.2 Notation. To avoid confusion with the notation for a point and an interval, we use the symbol $\langle v_1, v_2 \rangle$ for an ordered pair that represents a vector in **component form**. Lowercase letters are used to represent the components.

Definition 4.4.3 Position Vector. A vector $\mathbf{v}$ with initial point at $(0, 0)$ and terminal point at (a, b) is called the **position vector** and can be written as

$$\mathbf{v} = \langle a - 0, b - 0 \rangle = \langle a, b \rangle.$$

$\Diamond$

Remark 4.4.4 The position vector represents the terminal point of any vector **v** when its initial point is at the origin. Essentially, it is **v** translated so that its initial point is at the origin. This translation does not change the magnitude or direction, making the two vectors equivalent. Thus, we can refer to both vectors as **v**. In other words, the terminal point of a vector starting at the origin is defined by its horizontal and vertical components.

Example 4.4.5 Find the position vector of the vector that goes from $P(2, -1)$ to $Q(5, 3)$.

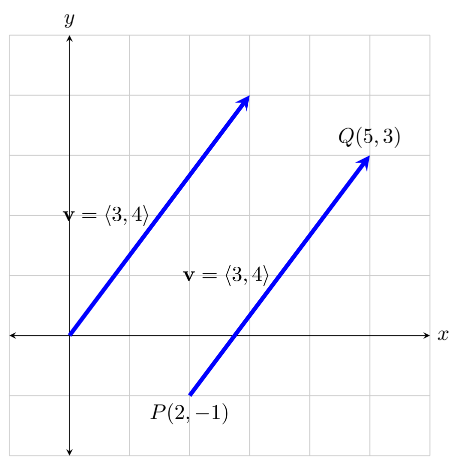
Solution. From Definition 4.4.1, our original vector can be written as:

$$\mathbf{v} = \overrightarrow{PQ} = \langle 5 - 2, 3 - (-1) \rangle = \langle 3, 4 \rangle.$$

Then by Definition 4.4.3, the position vector is:

$$\mathbf{v} = \langle 3, 4 \rangle.$$

This vector starts at the origin with 3 as its horizontal component and 4 as its vertical component, as illustrated in the figure below.



$\square$

Definition 4.4.6 Zero Vector. The **zero vector** is the vector $\mathbf{0} = \langle 0, 0 \rangle$ $\Diamond$

Definition 4.4.7 Magnitude of a Vector. Given a vector $\mathbf{v} = \langle v_x, v_y \rangle$, its **magnitude** or **length** is

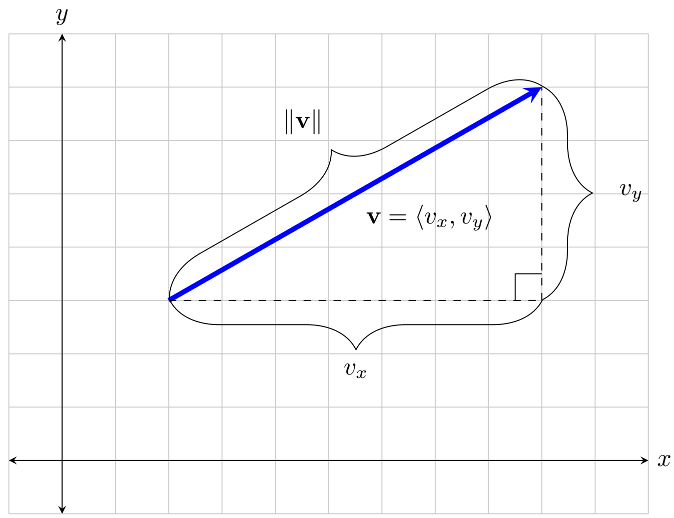
$$\|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2}.$$

$\Diamond$

Remark 4.4.8 This formula is derived from the Pythagorean Theorem or the distance formula where the square of the magnitude equals the square of the horizontal component plus the square of the vertical component:

$$\|\mathbf{v}\|^2 = v_x^2 + v_y^2.$$

This relationship can be visually understood using the Pythagorean Theorem, as illustrated in the following figure.



Example 4.4.9 Find the magnitude of each vector.

(a) $\mathbf{u} = \langle 10, -5 \rangle$

Solution.

$$\|\mathbf{u}\| = \sqrt{10^2 + (-5)^2} = \sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}$$

(b) $\mathbf{v} = \langle -2, 7 \rangle$

Solution.

$$\|\mathbf{v}\| = \sqrt{(-2)^2 + 7^2} = \sqrt{4 + 49} = \sqrt{53}$$

(c) $\mathbf{w} = \langle \frac{12}{13}, \frac{5}{13} \rangle$

Solution.

$$\|\mathbf{w}\| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = \sqrt{\frac{144}{169} + \frac{25}{169}} = \sqrt{\frac{169}{169}} = 1$$

□

Definition 4.4.10 Unit Vector. A vector $\mathbf{u}$ with magnitude 1, $\|\mathbf{u}\| = 1$, is called a **unit vector** ◊

Definition 4.4.11 Algebraic operations of vectors. If $\mathbf{u} = \langle u_x, u_y \rangle$, $\mathbf{v} = \langle v_x, v_y \rangle$, and c is a scalar, then

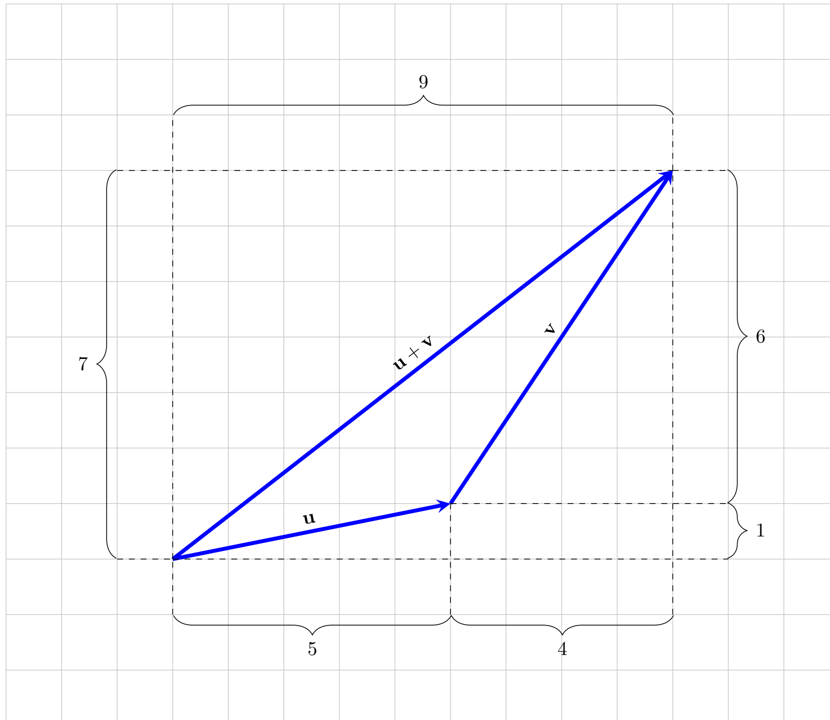
$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_x + v_x, u_y + v_y \rangle \\ \mathbf{u} - \mathbf{v} &= \langle u_x - v_x, u_y - v_y \rangle \\ c\mathbf{u} &= \langle cu_x, cu_y \rangle. \end{aligned}$$

◊

Example 4.4.12 Let $\mathbf{u} = \langle 5, 1 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$. Compute the following:

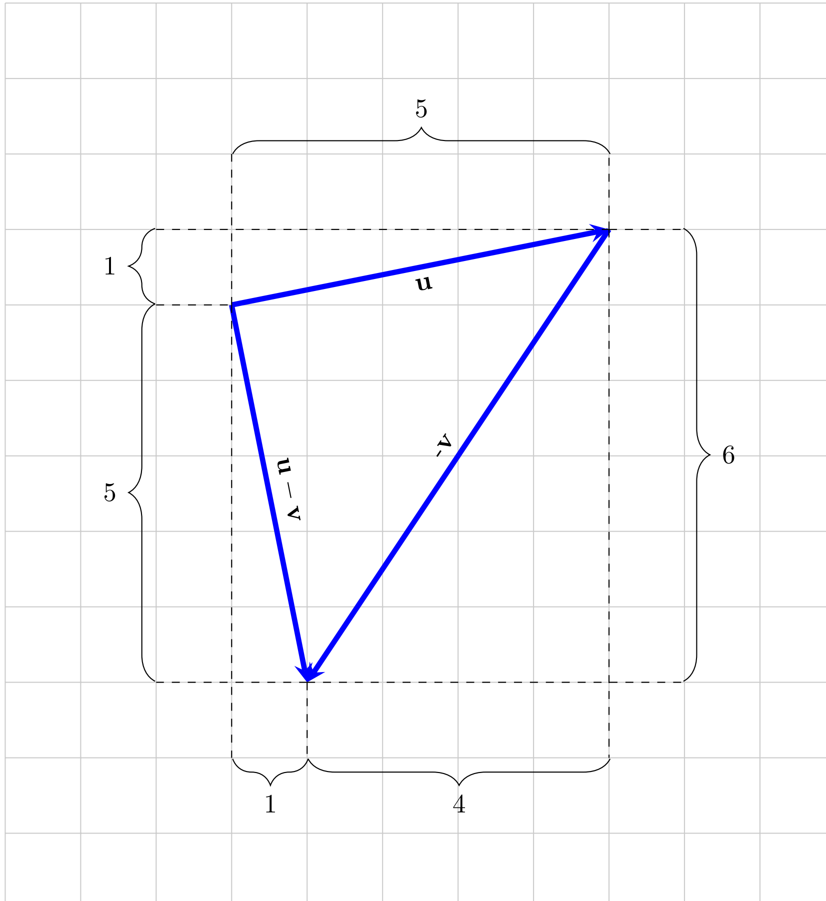
(a) $\mathbf{u} + \mathbf{v}$

Solution. $\mathbf{u} + \mathbf{v} = \langle 5, 1 \rangle + \langle 4, 6 \rangle = \langle 5 + 4, 1 + 6 \rangle = \langle 9, 7 \rangle$



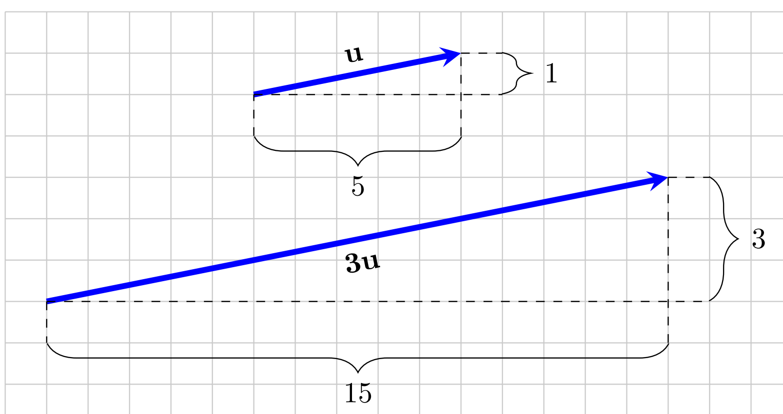
(b) $\mathbf{u} - \mathbf{v}$

Solution. $\mathbf{u} - \mathbf{v} = \langle 5, 1\rangle - \langle 4, 6\rangle = \langle 5 - 4, 1 - 6\rangle = \langle 1, -5\rangle$



(c) $3\mathbf{u}$

Solution. $3\mathbf{u} = 3\langle 5, 1\rangle = \langle 3\cdot 5, 3\cdot 1\rangle = \langle 15, 3\rangle$



□

4.4.2 Properties of Addition, Length, and Scalar Multiplication

Definition 4.4.13 Properties of Vectors. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, $\mathbf{0}$ is the zero vector, and c and d are scalars, then:

Vector Addition

- $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
- $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
- $\mathbf{u}+\mathbf{0}=\mathbf{u}$
- $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Length of a Vector

- $\|c\mathbf{u}\|=|c|\|\mathbf{u}\|$

Scalar Multiplication

- $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
- $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
- $(cd)\mathbf{u}=c(d\mathbf{u})=d(c\mathbf{u})$
- $1\mathbf{u}=\mathbf{u}$
- $0\mathbf{u}=\mathbf{0}$
- $c\mathbf{0}=\mathbf{0}$

◊

4.4.3 Finding a Unit Vector

Sometimes it is useful to look at problems involving vectors in terms of a vector in the same direction, but with a magnitude of one. Recall that a unit vector $\mathbf{u}$ is a vector whose length is one, $\|\mathbf{u}\|=1$. To find a unit vector in the same direction, we will need to multiply that vector by the reciprocal of its length or magnitude.

Definition 4.4.14 Unit Vector That Has the Same Direction as v. Let $\mathbf{v}$ be any vector. The vector

$$\mathbf{u}=\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is a **unit vector** that has the same direction as $\mathbf{v}$. ◊

Example 4.4.15 Find the unit vector that has the same direction as $\mathbf{v}=\langle 10,-5\rangle$.

Solution. From Example 4.4.9, we know $\|\mathbf{v}\|=5\sqrt{5}$. Then

$$\begin{aligned}\mathbf{u}&=\frac{\mathbf{v}}{\|\mathbf{v}\|}\\&=\frac{\langle 10,-5\rangle}{5\sqrt{5}}\\&=\left\langle\frac{10}{5\sqrt{5}},-\frac{5}{5\sqrt{5}}\right\rangle\end{aligned}$$

$$= \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle.$$

We can verify that this is in fact a unit vector by computing the magnitude:

$$\begin{aligned}
\|\mathbf{u}\| &= \left\| \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle \right\| \\
&= \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(-\frac{1}{\sqrt{5}}\right)^2} \\
&= \sqrt{\frac{4}{5} + \frac{1}{5}} \\
&= 1.
\end{aligned}$$

□

Two useful unit vectors are **i** and **j**. The vector **i** represents the unit vector whose direction is along the positive x-axis and the vector **j** represents the unit vector whose direction is along the positive y-axis.

Definition 4.4.16 i and j. The unit vectors **i** and **j** are defined as

$$\mathbf{i} = \langle 1, 0 \rangle, \quad \mathbf{j} = \langle 0, 1 \rangle.$$

◊

We can now represent any vector using the unit vectors **i** and **j**.

Definition 4.4.17 Horizontal and Vertical Components of a Vector. Let **v** be any vector. We can express **v** in terms of its **horizontal** and **vertical components**:

$$\mathbf{v} = \langle v_x, v_y \rangle = v_x\mathbf{i} + v_y\mathbf{j}.$$

◊

4.4.4 Vector Components and Direction

Recall that a vector is composed of a direction and magnitude. Earlier in this section we learned how to calculate the magnitude. Now we will discuss how to calculate the direction.

Definition 4.4.18 Let $\mathbf{v} = \langle v_x, v_y \rangle$ be a vector. The angle θ represents the **direction** of **v** and is the smallest positive angle in standard position formed by the positive x-axis and **v** ($0° \leq \theta < 360°$). ◊

Remark 4.4.19 Finding Direction of a Vector. Recall that

$$\tan\theta = \frac{v_y}{v_x}.$$

Since $-90° < \tan^{-1}\left(\frac{v_y}{v_x}\right) < 90°$, the inverse tangent function returns an angle in Quadrants I or IV. To determine the correct direction θ, consider the quadrant in which the vector lies and make the appropriate adjustments:

- If θ is in Quadrant I, then $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$.
- If θ is in Quadrant II or III, then $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) + 180°$.
- If θ is in Quadrant IV, then $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) + 360°$.

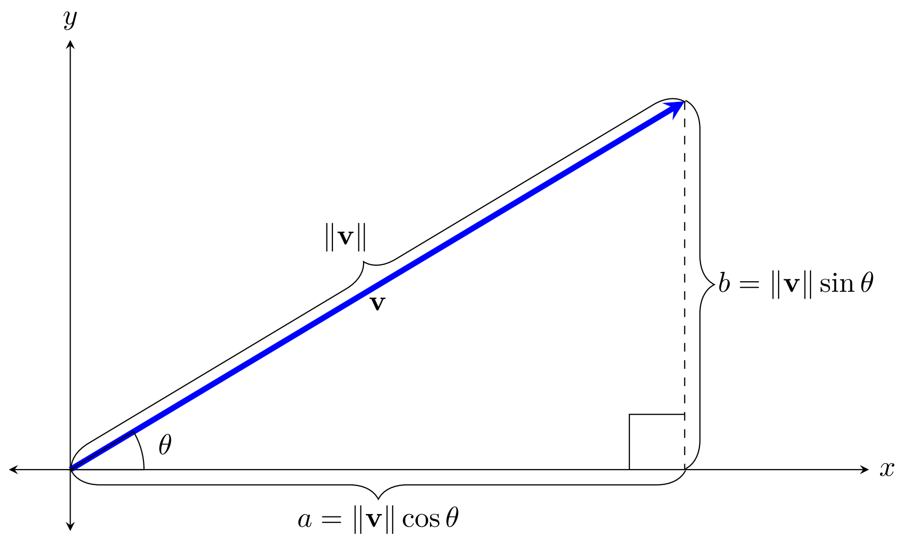
We now have all the information needed to write a vector.

Definition 4.4.20 Finding Horizontal and Vertical Components of Vectors from Magnitude and Direction. For vector $\mathbf{v}$ with magnitude $\|\mathbf{v}\|$ and direction θ, we can use Right Triangle Trigonometry to solve for the horizontal and vertical components, denoted as v_x and v_y, respectively:

$$v_x = \|\mathbf{v}\| \cos\theta, \quad v_y = \|\mathbf{v}\| \sin\theta.$$

This allows us to express $\mathbf{v}$ as:

$$\mathbf{v} = \|\mathbf{v}\| \cos\theta \mathbf{i} + \|\mathbf{v}\| \sin\theta \mathbf{j} = \langle \|\mathbf{v}\| \cos\theta, \|\mathbf{v}\| \sin\theta \rangle.$$



◊

4.4.5 Velocity

Vectors can be used to represent the velocity of a moving canoe. These **velocity vectors** have a direction and magnitude.

Example 4.4.21 Wind is blowing from the house Nā Leo Hoʻolua (N22.5°W) at 10 knots. Write the wind velocity as a vector $\mathbf{v}$.

Solution. First, we note that since the velocity is 10 knots, we have $\|\mathbf{v}\| = 10$. Next, we need to find the direction. Since θ is measured from the positive x-axis, by Definition 4.4.20 we see that

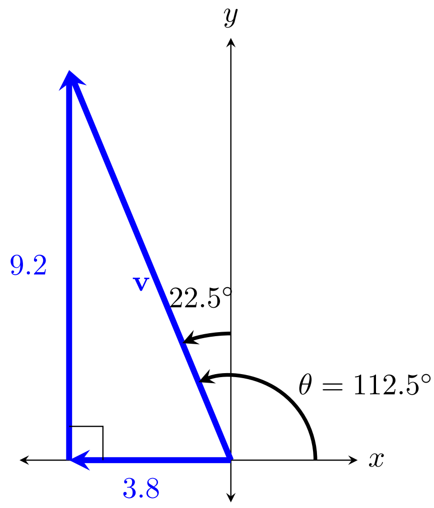
$$\theta = 90^\circ + 22.5^\circ = 112.5^\circ.$$

To find the vector, will need the horizontal and vertical components:

$$\begin{aligned} v_x =& \|\mathbf{v}\| \cos\theta = 10\cos 112.5^\circ \approx -3.8 \\ v_y =& \|\mathbf{v}\| \sin\theta = 10\sin 112.5^\circ \approx 9.2. \end{aligned}$$

Thus

$$\mathbf{v} \approx -3.8\mathbf{i} + 9.2\mathbf{j} = \langle -3.8, 9.2 \rangle.$$

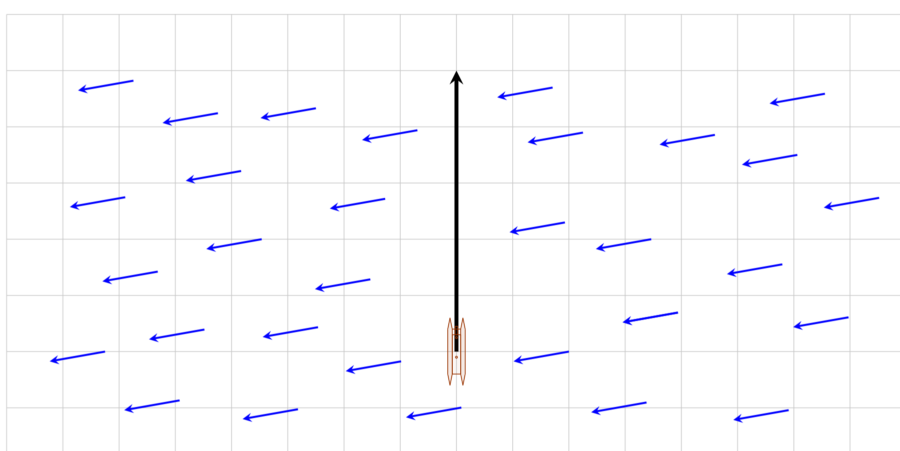


□

Example 4.4.22 Calculating Canoe Velocity with Current Drift. The vaka Marumaru Atua sets sail on a northward voyage from Rarotonga to Hawaiʻi, maintaining a steady speed of 5 knots through the water. However, a 1-knot current flows in the direction of Lā Kona (a heading of 260 degrees). This current's influence is like walking across a moving floor—no matter how steadily the vaka sails north, the entire ocean beneath it is shifting, altering its actual path. Similarly, Marumaru Atua navigates while the water, in the form of a current, also moves. In sailing, the direction and speed at which the current is pushing the vaka are referred to as **set** and **drift**, respectively.

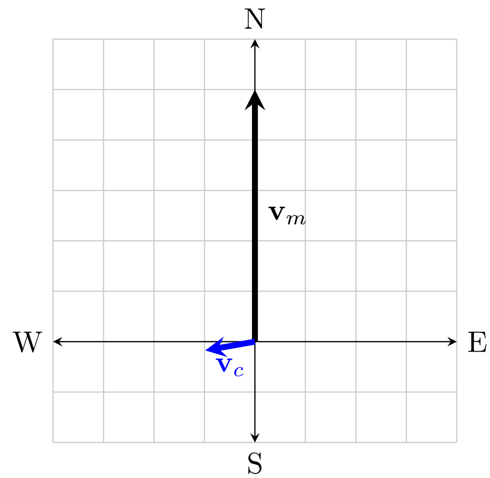
The actual velocity of Marumaru Atua is the resultant of the vaka's velocity and the current's velocity. This resultant velocity vector denotes the vaka's speed and direction relative to fixed objects on Earth, influenced by the current's impact on Marumaru Atua. The magnitude and direction of this velocity vector are represented by the **speed over ground (SOG)** and **course over ground (COG)**, respectively. SOG represents the speed of the vaka relative to fixed objects, accounting for both the vaka's speed through the water and the current's speed and direction. COG indicates the direction of the vaka's motion over the Earth's surface.

Understanding the difference between the velocity over water and the velocity relative to fixed objects illustrates the influence set and drift have on the vaka's course and speed over ground. Voyagers need to account for set and drift when navigating to ensure they reach their intended destination accurately and safely.



(a) Express Marumaru Atua's velocity vector, represented as $\mathbf{v}_m$, and the velocity vector of the current, denoted as $\mathbf{v}_c$, in terms of their horizontal and vertical components. Round the answer to two decimal places.

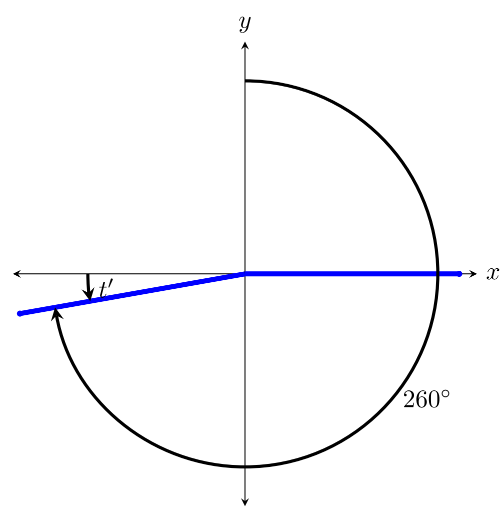
Solution. First, we draw the velocity vectors for Marumaru Atua and the current:



Since Marumaru Atua is sailing north at 5 knots, the velocity is:

$$\mathbf{v}_m = \langle 0, 5 \rangle.$$

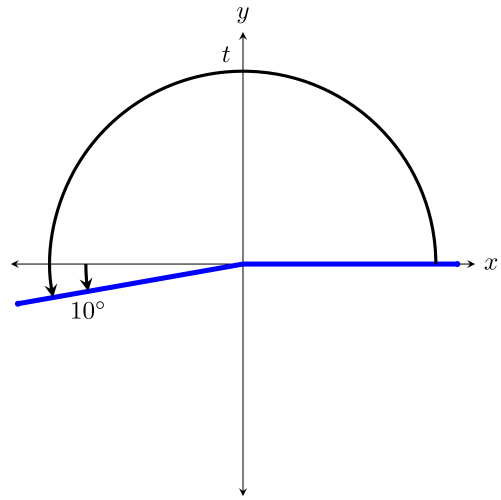
To find the vector for the current, we need to know t, the angle of the velocity vector in standard position (see Definition 1.2.8). We begin by drawing the heading angle of 260° and its reference angle, t'.



We can see from the figure that $260° + t' = 270°$, thus we get our reference angle as

$$t' = 10°.$$

We now draw our angle, t, in standard position and the reference angle, t':



From Remark 1.5.9, we have

$$t = 180^\circ + 10^\circ = 190^\circ.$$

Thus, by Definition 4.4.20 the velocity for the current is

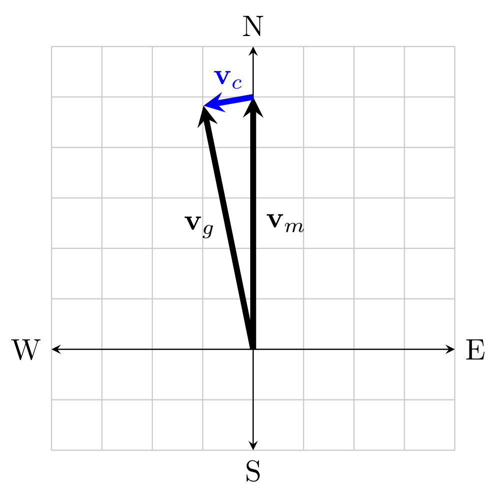
$$\mathbf{v}_c = \langle 1\cos 190^\circ, 1\sin 190^\circ \rangle \approx \langle -0.98, -0.17 \rangle.$$

(b) Find $\mathbf{v}_g$, the velocity of Marumaru Atua relative to fixed objects on the ground.

Solution. The velocity relative to fixed objects on the ground is the sum of the two vectors:

$$\begin{aligned} \mathbf{v}_g = \mathbf{v}_m + \mathbf{v}_c &= \langle 0, 5 \rangle + \langle -0.98, -0.17 \rangle \\ &= \langle -0.98, 5 - 0.17 \rangle \\ &= \langle -0.98, 4.83 \rangle. \end{aligned}$$

This is visually demonstrated in the figure below:



(c) Find the speed over ground (SOG) and the course over ground (COG) of Marumaru Atua, rounded to one decimal place.

Solution. The actual speed of Marumaru Atua is simply the magnitude of the velocity over ground:

$$\|\mathbf{v}_g\| = \|\langle -0.98, 4.83\rangle\| = \sqrt{(-0.98)^2 + (4.83^2)} \approx 4.9 \text{ knots.}$$

Thus, although the vaka sails through the water at 5 knots, its forward progress is slightly slower with a speed over ground at 4.9 knots.

To find the course over ground, we will use Remark 4.4.19 to find direction of the velocity:

$$\tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{4.83}{-0.98} \approx -78.5^\circ.$$

Since our velocity vector is in Quadrant II, our course over ground is

$$\theta = -78.5^\circ + 180^\circ = 101.5^\circ.$$

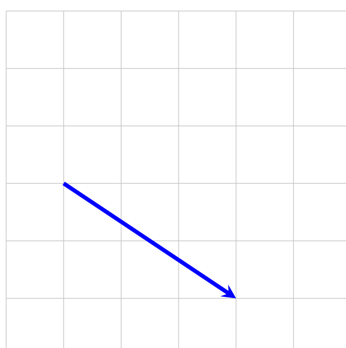
□

4.4.6 Exercises

Exercise Group. Draw the vector represented by the given components.

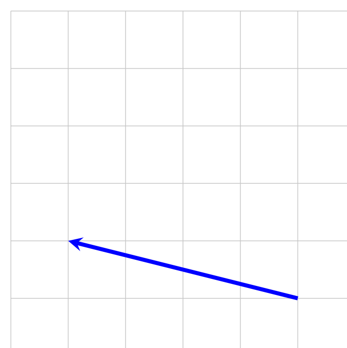
1. $3\mathbf{i} - 2\mathbf{j}$

Answer.



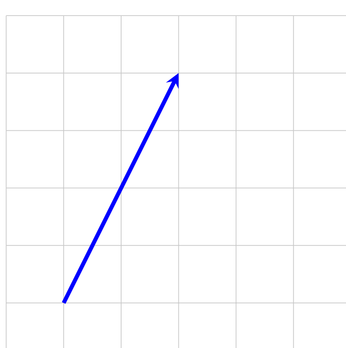
2. $-4\mathbf{i} + \mathbf{j}$

Answer.



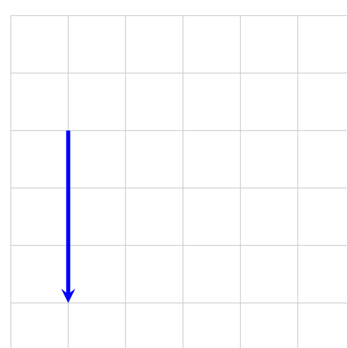
3. $2\mathbf{i} + 4\mathbf{j}$

Answer.



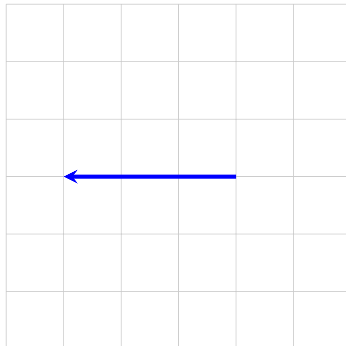
4. $-3\mathbf{j}$

Answer.



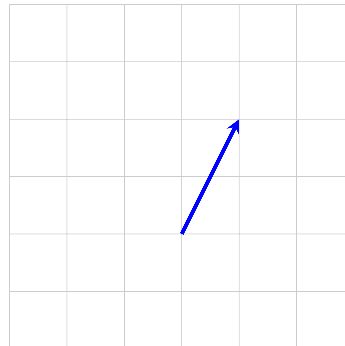
5. $\langle -3, 0\rangle$

Answer.



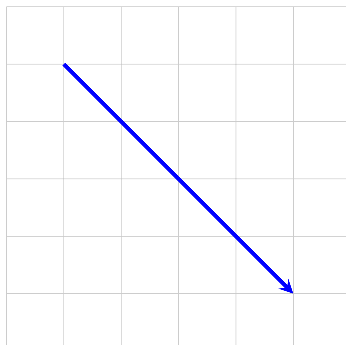
6. $\langle 1, 2\rangle$

Answer.



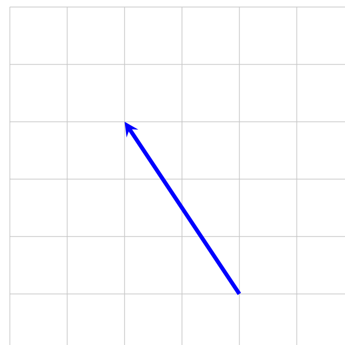
7. $\langle 4, -4\rangle$

Answer.



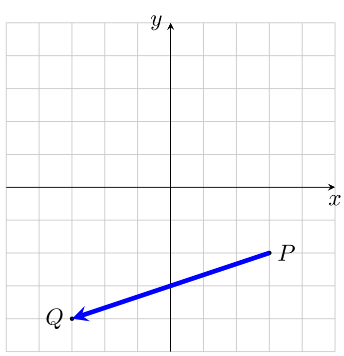
8. $\langle -2, 3\rangle$

Answer.



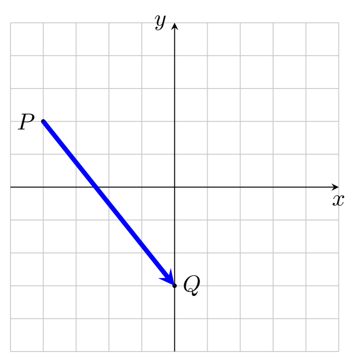
Exercise Group. For each of the following, find the vector with initial point at P and terminal point at Q. Express your answer in form $\langle a, b\rangle$.

9.



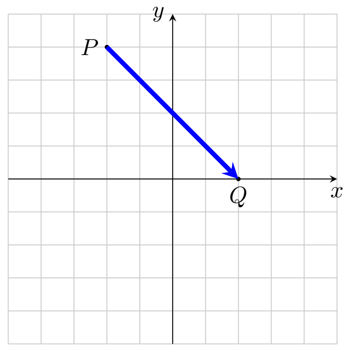
Answer. $\langle -6, -2\rangle$

10.



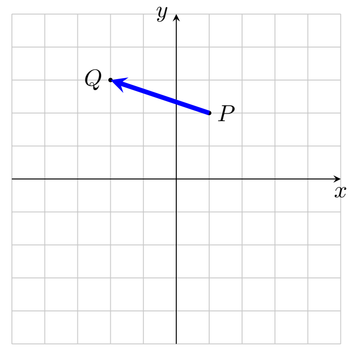
Answer. $\langle 4, -5\rangle$

11.



Answer. $\langle 4, -4\rangle$

12.



Answer. $\langle -3, 1\rangle$

13. $P = (4, -6)$, $Q = (10, 1)$

Answer. $\langle 6, 7\rangle$

14. $P = (-8, -5)$, $Q = (5, 8)$

Answer. $\langle 13, 13\rangle$

15. $P = (2, 0)$, $Q = (-7, -3)$

Answer. $\langle -9, -3\rangle$

16. $P = (1, 6)$, $Q = (-4, -2)$

Answer. $\langle -5, -8\rangle$

17. $P = (-6, 4)$, $Q = (7, -9)$

Answer. $\langle 13, -13\rangle$

18. $P = (-1, 5)$, $Q = (3, -8)$

Answer. $\langle 4, -13\rangle$

19. $P = (9, -1)$, $Q = (-10, 10)$

Answer. $\langle -19, 11\rangle$

20. $P = (-7, 6)$, $Q = (4, -3)$

Answer. $\langle 11, -9\rangle$

Exercise Group. Express each vector in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

21. $\langle -1, 4\rangle$

Answer. $-\mathbf{i} + 4\mathbf{j}$

22. $\langle 2, -1\rangle$

Answer. $2\mathbf{i} - \mathbf{j}$

23. $\langle 2, 5\rangle$

Answer. $2\mathbf{i} + 5\mathbf{j}$

24. $\langle 3, 3\rangle$

Answer. $3\mathbf{i} + 3\mathbf{j}$

Exercise Group. Express each vector in the form $\langle a, b\rangle$.

25. $-4\mathbf{i} - 2\mathbf{j}$

Answer. $\langle -4, -2\rangle$

26. $\mathbf{i} - 3\mathbf{j}$

Answer. $\langle 1, -3\rangle$

27. $-2\mathbf{i} + \mathbf{j}$

Answer. $\langle -2, 1\rangle$

28. $4\mathbf{i}$

Answer. $\langle 4, 0\rangle$

Exercise Group. For given vectors $\mathbf{u}$ and $\mathbf{v}$, find $2\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $3\mathbf{u}+2\mathbf{v}$.

29. $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle 2, 0\rangle$

Answer. $2\mathbf{u} = \langle 2, 4\rangle$,
$\mathbf{u}+\mathbf{v} = \langle 3, 2\rangle$, $\mathbf{u}-\mathbf{v} = \langle -1, 2\rangle$,
$3\mathbf{u}+2\mathbf{v} = \langle 7, 6\rangle$

30. $\mathbf{u} = \langle 5, 1\rangle$, $\mathbf{v} = \langle -3, -2\rangle$

Answer. $2\mathbf{u} = \langle 10, 2\rangle$,
$\mathbf{u}+\mathbf{v} = \langle 2, -1\rangle$, $\mathbf{u}-\mathbf{v} = \langle 8, 3\rangle$,
$3\mathbf{u}+2\mathbf{v} = \langle 9, -1\rangle$

31. $\mathbf{u} = \langle -2, 4\rangle$, $\mathbf{v} = \langle 0, 4\rangle$

Answer. $2\mathbf{u} = \langle -4, 8\rangle$,
$\mathbf{u}+\mathbf{v} = \langle -2, 8\rangle$,
$\mathbf{u}-\mathbf{v} = \langle -2, 0\rangle$,
$3\mathbf{u}+2\mathbf{v} = \langle -6, 20\rangle$

32. $\mathbf{u} = \langle -1, -5\rangle$, $\mathbf{v} = \langle 1, 2\rangle$

Answer. $2\mathbf{u} = \langle -2, -10\rangle$,
$\mathbf{u}+\mathbf{v} = \langle 0, -3\rangle$,
$\mathbf{u}-\mathbf{v} = \langle -2, -7\rangle$,
$3\mathbf{u}+2\mathbf{v} = \langle -1, -11\rangle$

33. $\mathbf{u} = \langle 3, -1\rangle$, $\mathbf{v} = \langle -4, 4\rangle$

Answer. $2\mathbf{u} = \langle 6, -2\rangle$,
$\mathbf{u} + \mathbf{v} = \langle -1, 3\rangle$,
$\mathbf{u} - \mathbf{v} = \langle 7, -5\rangle$,
$3\mathbf{u} + 2\mathbf{v} = \langle 1, 5\rangle$

34. $\mathbf{u} = \langle -2, -2\rangle$, $\mathbf{v} = \langle 1, 3\rangle$

Answer. $2\mathbf{u} = \langle -4, -4\rangle$,
$\mathbf{u} + \mathbf{v} = \langle -1, 1\rangle$,
$\mathbf{u} - \mathbf{v} = \langle -3, -5\rangle$,
$3\mathbf{u} + 2\mathbf{v} = \langle -4, 0\rangle$

Exercise Group. A fundamental property in Euclidean geometry is the **Triangle Inequality**, which states that for any triangle, the sum of the lengths of any two sides of a triangle is greater than or equal to the length of the remaining side. This property can be illustrated using vectors, where combining vectors $\mathbf{u}$ and $\mathbf{v}$ results in a triangle with the third side represented by $\mathbf{u} + \mathbf{v}$, as shown in Figure 4.3.14. The Triangle Inequality is expressed as $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$.

To demonstrate the Triangle Inequality with given vectors $\mathbf{u}$ and $\mathbf{v}$, calculate the following magnitudes rounded to the nearest tenth: $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u}\| + \|\mathbf{v}\|$, and $\|\mathbf{u} + \mathbf{v}\|$. Note that to calculate $\|\mathbf{u} + \mathbf{v}\|$, first find the vector $\mathbf{u} + \mathbf{v}$

35. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

Answer. $\|\mathbf{u}\| = 2.8$,
$\|\mathbf{v}\| = 5$, $\|\mathbf{u}\| + \|\mathbf{v}\| = 7.8$,
$\|\mathbf{u} + \mathbf{v}\| = 5.4$

36. $\mathbf{u} = 5\mathbf{i} - \mathbf{j}$, $\mathbf{v} = -3\mathbf{i}$

Answer. $\|\mathbf{u}\| = 5.1$,
$\|\mathbf{v}\| = 3$, $\|\mathbf{u}\| + \|\mathbf{v}\| = 8.1$,
$\|\mathbf{u} + \mathbf{v}\| = 2.2$

37. $\mathbf{u} = \langle -2, 4\rangle$, $\mathbf{v} = \langle 3, -4\rangle$

Answer. $\|\mathbf{u}\| = 4.5$,
$\|\mathbf{v}\| = 5$, $\|\mathbf{u}\| + \|\mathbf{v}\| = 9.5$,
$\|\mathbf{u} + \mathbf{v}\| = 1$

38. $\mathbf{u} = \langle 0, 2\rangle$, $\mathbf{v} = \langle 1, 3\rangle$

Answer. $\|\mathbf{u}\| = 2$,
$\|\mathbf{v}\| = 3.2$, $\|\mathbf{u} + \mathbf{v}\| = 5.1$,
$\|\mathbf{u}\| + \|\mathbf{v}\| = 5.2$

39. $\mathbf{u} = \langle -4, -1\rangle$, $\mathbf{v} = \langle 4, -3\rangle$

Answer. $\|\mathbf{u}\| = 4.1$,
$\|\mathbf{v}\| = 5$, $\|\mathbf{u}\| + \|\mathbf{v}\| = 9.1$,
$\|\mathbf{u} + \mathbf{v}\| = 4$

40. $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle 4, 2\rangle$

Answer. $\|\mathbf{u}\| = 2.2$,
$\|\mathbf{v}\| = 4.5$, $\|\mathbf{u}\| + \|\mathbf{v}\| = 6.7$,
$\|\mathbf{u} + \mathbf{v}\| = 6.7$

Exercise Group. Given the vector $\mathbf{v}$, find the unit vector $\mathbf{u}$ in the same direction. Verify that $\|\mathbf{u}\| = 1$.

41. $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$

Answer. $\mathbf{u} = -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{j}$;
$\|\mathbf{u}\| = 1$

42. $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$

Answer. $\mathbf{u} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$;
$\|\mathbf{u}\| = 1$

43. $\mathbf{v} = \langle 2, -5\rangle$

Answer. $\mathbf{u} = \left\langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right\rangle$;
$\|\mathbf{u}\| = 1$

44. $\mathbf{v} = \langle 3, -2\rangle$

Answer. $\mathbf{u} = \left\langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right\rangle$;
$\|\mathbf{u}\| = 1$

45. $\mathbf{v} = \langle 0, -4\rangle$

Answer. $\mathbf{u} = \langle 0, -1\rangle$;
$\|\mathbf{u}\| = 1$

46. $\mathbf{v} = \langle 5, 1\rangle$

Answer. $\mathbf{u} = \left\langle \frac{5}{\sqrt{26}}, \frac{1}{\sqrt{26}} \right\rangle$;
$\|\mathbf{u}\| = 1$

Exercise Group. For each problem, the magnitude and direction of vector $\mathbf{v}$ are given. Find the horizontal and vertical components of the vector, v_x and v_y, respectively, and write the vector in the form $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$.

47. $\|\mathbf{v}\| = 2$; $\theta = 135°$

Answer. $\mathbf{v} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

48. $\|\mathbf{v}\| = 6$; $\theta = 300°$

Answer. $\mathbf{v} = 3\mathbf{i} - 3\sqrt{3}\mathbf{j}$

49. $\|\mathbf{v}\| = 4$; $\theta = 210°$

Answer. $\mathbf{v} = -2\sqrt{3}\mathbf{i} - 2\mathbf{j}$

50. $\|\mathbf{v}\| = 3$; $\theta = 30°$

Answer. $\mathbf{v} = \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$

51. $\|\mathbf{v}\| = 5$; $\theta = 120°$

Answer. $\mathbf{v} = -\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}$

52. $\|\mathbf{v}\| = 2$; $\theta = 315°$

Answer. $\mathbf{v} = \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$

Exercise Group. Find the magnitude and direction angle of vector $\mathbf{v}$. Round the answers to one decimal place and express angles in degrees.

53. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}$

Answer. $\|\mathbf{v}\| = 2.8$; $\theta = 45°$

54. $\mathbf{v} = -3\mathbf{i} + 7\mathbf{j}$

Answer. $\|\mathbf{v}\| = 7.6$; $\theta = 113.2°$

55. $\mathbf{v} = \langle 4, -1\rangle$

Answer. $\|\mathbf{v}\| = 4.1$; $\theta = 346.0°$

56. $\mathbf{v} = \langle 3, 4\rangle$

Answer. $\|\mathbf{v}\| = 5$; $\theta = 53.1°$

57. $\mathbf{v} = \langle 0, -3\rangle$

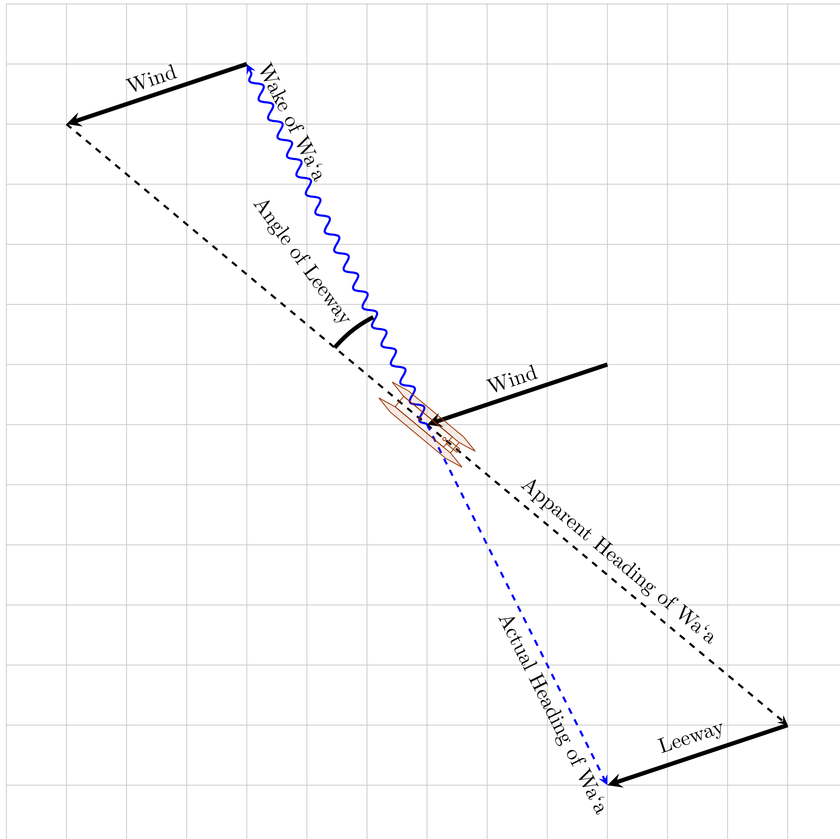
Answer. $\|\mathbf{v}\| = 3$; $\theta = 270°$

58. $\mathbf{v} = \langle -\sqrt{3}, -1\rangle$

Answer. $\|\mathbf{v}\| = 2$; $\theta = 210°$

59. Leeway. When a wa‘a sails on the ocean, it rarely moves precisely in the direction it’s pointed. One reason for this is that crosswind can push the wa‘a off its course. The angle of displacement between the apparent heading of the wa‘a and the direction the wa‘a is actually traveling through the water is referred to as **leeway**. Using vector addition, we can determine where the wa‘a will actually travel. However, on the wa‘a, the navigator can determine the leeway by observing the angle between the wake behind the wa‘a and the apparent direction the wa‘a is pointed towards.

In order to compensate for the wind, the navigator must steer the wa‘a into the wind by the same angle as the leeway angle. For example, if the wa‘a needs to sail in the house Manu Malanai and the wind is pushing the wa‘a one house further south, the wa‘a will move in the house Nālani Malanai. To maintain the course in the house Manu Malanai, the navigator must then point the wa‘a one house north with an apparent heading in the house of Noio Malanai in order for the actual heading to be in the house of Manu Malanai.



If the apparent heading is represented by the vector $\langle 6, -5\rangle$ and the leeway is represented by the vector $\langle -3, -1\rangle$, calculate the vector for the actual heading.

Answer. $\langle 3, -6\rangle$

Appendix A

An Introduction to the Canoes in This Book

Here we will introduce the canoes ffeatured in this book.

A.1 Alingano Maisu

The Alingano Maisu is a double-hulled voyaging canoe built in Kawaihae in 2007 by members of Na Kalai Wa'a Moku o Hawai'i as a gift to Mau Piailug and the people of Satawal. It is the sister canoe to Makali'i.

A.2 Faʻafaite

Faʻafaite is a double-hulled voyaging canoe built in 2009. It is one of eight vaka moana built for the Okeanos Foundation for the Sea. It is owned by the Faʻafaite - Tahiti Voyaging Society.

A.3 Gaualofa

Gaualofa is a double-hulled voyaging canoe built in 2012. It is one of eight vaka moana built for the Okeanos Foundation for the Sea. It is owned by the Samoan Voyaging Society.

A.4 Haunui

Haunui is a double-hulled voyaging canoe that was relaunched in 2011. It is owned by Te Toki Voyaging Trust.

A.5 Hawai‘iloa

Hawai‘iloa is a double-hulled voyaging canoe built in 2009.

A.6 Hinemoana

Hinemoana is a double-hulled voyaging canoe built in 2009.

A.7 Hōkūleʻa

Hōkūleʻa is a double-hulled voyaging canoe built in 2009.

A.8 Kānehūnāmoku

Kānehūnāmoku is a double-hulled voyaging canoe built in 2009.

A.9 Mānaiakalani

Mānaiakalani is a double-hulled voyaging canoe built in 2009.

A.10 Marumaru Atua

Marumaru Atua is a double-hulled voyaging canoe built in 2009.

A.11 Paikea

Paikea is a double-hulled voyaging canoe built in 2009.

A.12 Te Matau a Māui

Faafaite is a double-hulled voyaging canoe built in 2009.

A.13 Uto ni Yalo

Uto ni Yalo is a double-hulled voyaging canoe built in 2008 for the Uto ni Yalo Trust

A.14 Vaka Moana

Vaka Moana is a double-hulled voyaging canoe built in 2009.

Bibliography

Glossary (Math)

amplitude. definition
period. definition

Glossary (Canoe and Wayfinding)

heading. The direction in which a canoe is pointed or moving, with true north corresponding to 0° and positive angles measured in a clockwise direction.

latitude sailing. A navigational technique in which a canoe sails north or south to reach the latitude of its destination before turning east or west. Because stars move in an arc from east to west, they do not provide a precise measure of longitude. However, their positions and movements shift with latitude. By tracking these changes, navigators can determine when they have reached the correct latitude, allowing them to adjust course toward their destination.

waʻa kaulua. A double-hulled voyaging canoe used by Native Hawaiians for long-distance sailing, traditionally navigated using wayfinding techniques, without the aid of modern instruments.

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